

ON SCHRÖDINGER MAPS

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ABSTRACT. We study the question of well-posedness of the Cauchy problem for Schrödinger maps from $\mathbb{R}^1 \times \mathbb{R}^2$ to the sphere \mathbb{S}^2 or to \mathbb{H}^2 , the hyperbolic space. The idea is to choose an appropriate gauge change so that the derivatives of the map will satisfy a certain nonlinear Schrödinger system of equations and then study this modified Schrödinger map system (MSM). We then prove local well posedness of the Cauchy problem for the MSM with minimal regularity assumptions on the data and outline a method to derive well posedness of the Schrödinger map itself from it. In proving well posedness of the MSM, the heart of the matter is resolved by considering truly quatrilinear forms of weighted L^2 functions.

1. INTRODUCTION

The harmonic map equation between two Riemannian manifolds is one of the most studied equations in the modern geometric analysis. There are three evolution equations which are derived from the same geometric considerations. The best known one is the heat flow for harmonic maps, which was, in fact, used by Eells and Sampson in one of the first papers on harmonic maps. This flow equation has been successfully studied by methods which in spirit depend on the same geometric ideas used in the elliptic theory of harmonic mappings.

In the last decade, the wave equation version, the wave map equation, has been studied by a number of mathematicians. The work of Klainerman is probably the best known, and the recent work of Tao [9] [10] is very promising. The methods are quite different in spirit from the elliptic theory, and with the exception of the classical work on the equation in $1+1$ dimensions and some specialized work, use little in the spirit of gauge theoretic geometric methods. In this paper, we obtain estimates (which are sufficient to give estimates down to but not including the critical energy space) for the Schrödinger map equation in the special case from \mathbb{R}^2 to the sphere \mathbb{S}^2 (or to \mathbb{H}^2 , the hyperbolic space). The historical development of the theory of this equation demonstrates the need for some geometric insight.

The general formulation of the Schrödinger equation, which we will not need here, arises from writing the heat flow equation for harmonic maps into a Kähler manifold X . Let

$$s : \mathbb{R}^n \rightarrow X.$$

The heat flow is then described by

$$\frac{ds}{dt} = \nabla_s * ds.$$

Since the Kähler manifold X has an action by a complex structure $J(s)$ in the tangent bundle, the Schrödinger map equation can be written

$$J(s) \frac{ds}{dt} = \nabla_s * ds.$$

However the equation we are treating arises in a more natural fashion from the Landau-Lifschitz equation for a macroscopic ferromagnetic continuum [6] for $s : \mathbb{R}^n \rightarrow \mathbb{S}^2$, by considering \mathbb{S}^2 as embedded into \mathbb{R}^3 and

$$\frac{ds}{dt} = -s \times \Delta s.$$

To understand mathematically the one dimensional case, it is necessary to make a change of coordinates or gauge change, classically known as the Hasimoto transformation. A special gauge in the bundle $s^*T(\mathbb{S}^2)$ is chosen in which the covariant derivative in the space direction is the ordinary differentiation $\nabla_x = d/dx$. In these coordinates, we have

$$i\nabla_t u = \nabla_x^2 u = \frac{d^2 u}{dx^2},$$

where $u = ds/dx$ and $\nabla_t = d/dt + a_0$. However, the curvature R in the image is given by

$$[\nabla_x, \nabla_t] = \frac{d}{dx} a_0 = R(ds/dx, ds/dt) = R(u, i \frac{du}{dx}).$$

Since the curvature of X is constant at 1, some simple Kähler geometry gives

$$R(u, i \frac{du}{dx}) = \frac{i}{2} \frac{d|u|^2}{dx}$$

or

$$a_0 = \frac{i}{2} |u|^2.$$

In these coordinates, the equation becomes the usual integrable focusing non-linear Schrödinger equation. If we take \mathbb{H}^2 instead, we obtain the defocusing case with a change of sign.

Chang, Shatah and Uhlenbeck were able to handle the one dimensional case for arbitrary surfaces and the radially symmetric case $n = 2$ in the energy norm by an extension of this argument [3]. Our estimates follow those in spirit, although in two dimensions it is not possible to gauge away the derivative term completely.

We outline a proof of well-posedness in the coordinates we use. The coordinates we use are not the coordinates of the map, and we do not go into the technicalities of translating back and forth, primarily because the theory does not seem to be at this stage of development.

The plan of the paper is as follows. In Section 2 we give the coordinate change from a form of the Schrödinger map equation to \mathbb{S}^2 to the form in which we are able to make estimates. The origin of the estimates would be totally mysterious without this explanation. In Section 3 we state the fundamental estimates, which are cubic and quintic non-linearities. The one nonlinearity that contains the derivative is by far the hardest one to handle. We also state the basic estimate for initial data in H^ε (which corresponds to $H^{1+\varepsilon}$ for the map). We then include for convenience some estimates from [4] and [8], that are frequently used throughout the proof. The details of the proofs of estimates on various terms, which is the meat of the paper, are in Sections 5, 6, 7, 8. We use the mixed space-time Hilbert spaces $X_{s,b}$ as introduced by Bourgain for the non-linear Schrödinger equation without derivative non-linearities. These are well suited to study the low regularity behavior of these non-linear dispersive equations. It is interesting to note that at some stage, we need to use the mixed Lebesgue spaces $L_t^p L_x^q$ to handle the quintic nonlinearities. Our proof relies on and adapts from certain multilinear estimates recently obtained by Tao [8] and Colliander-Delort-Kenig-Staffilani [4]. The authors are appreciative of the clarity, breadth and availability of the work

of T. Tao, [8]. We note however that the heart of the matter is resolved by considering truly quadrilinear forms of weighted L^2 functions.

We believe that similar results must hold in all dimensions, with H^ϵ replaced by $H^{n/2-1+\epsilon}$. The sign of the curvature is not relevant to our equation, so the results hold for maps into the hyperbolic space \mathbb{H}^2 as well. In principle, it should not be difficult to extend the estimates to non-constant curvature surfaces, much as is done in the one-dimensional case ([3]). Since non-abelian gauge theory will be relevant for image manifolds of complex dimension larger than 1, the case of higher dimension in the target is much more difficult. Of course, we expect and hope that there are estimates which hold at the critical scaling regularity. It is worth noting that for the wave map equation, the spaces $X_{s,b}$, which we use are not adequate enough to handle the critical case, so we are not surprised that the estimates work down to but not at the critical case.

2. FORMULATION OF THE PROBLEM

The Schrödinger map equation for $\mathbb{R}^1 \times \mathbb{R}^n \rightarrow \mathbb{S}^2$ has a number of different descriptions, which are equivalent for smooth solutions. We describe this equation for all n but consider in the rest of the paper only $n = 2$. We start with a description in terms of the stereographic projection of $\mathbb{S}^2 \setminus \{N\} \rightarrow \mathbb{C}^1$ where N is the north pole. This is possibly the simplest for those unfamiliar with differential geometry. The estimates we obtain are in coordinate (gauge) system, which is dependent on the map, but independent of any coordinate choice.

Let $s : \mathbb{R}^n \rightarrow \mathbb{C}^1 \cup \{\infty\} = \mathbb{S}^2$. Then the energy of s is

$$E(s) = \frac{1}{2} \int_{\mathbb{R}^n} \sum_{j=1}^n \frac{|\partial s / \partial x_j|^2}{(1 + |s|^2)^2} (dx)^n.$$

A simple calculation shows that the Euler-Lagrange equations, or the equations for a harmonic map are

$$\begin{aligned} & \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\frac{\partial s}{\partial x_j} / (1 + |s|^2)^2 \right) + \\ & 2 \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\left| \frac{\partial s}{\partial x_j} \right|^2 / (1 + |s|^2)^3 \right) = 0. \end{aligned}$$

After a short computation, we find that this can be written as

$$\sum_j \nabla_j \frac{\partial s}{\partial x_j} = 0.$$

Here

$$\nabla_j = \frac{\partial}{\partial x_j} - 2 \left(\frac{\partial s}{\partial x_j} \bar{s} \right) / (1 + |s|^2),$$

is the covariant derivative corresponding to the pull-back of the Levi-Civita connection on the tangent plane $T(\mathbb{S}^2)$ by the map s .

The heat equation would be

$$\frac{\partial s}{\partial t} = \sum_j \nabla_j \frac{\partial s}{\partial x_j}.$$

The Schrödinger map equation is

$$\frac{\partial s}{\partial t} = \pm i \sum_j \nabla_j \frac{\partial s}{\partial x_j}.$$

We will change gauge in this equation from the coordinate frame of the stereographic projection to a normalized frame, and rotate the frame to put the pull-back covariant derivative ∇_j as near to $\frac{\partial}{\partial x_j}$ as possible. Since we will lose track of the map s during this process, we will need a set of consistency equations, which would be needed to recover the map s . So in addition to the equation

$$(1) \quad \frac{\partial s}{\partial t} = i \sum_j \nabla_j \frac{\partial s}{\partial x_j}.$$

we have two sets of consistency conditions:

$$(2) \quad \nabla_j \frac{\partial s}{\partial x_k} = \nabla_k \frac{\partial s}{\partial x_j} \quad \begin{array}{l} j = 0, 1, \dots, n \\ k = 1, \dots, n \end{array}$$

This can be computed and follows from the fact that the Levi-Civita connection on \mathbb{S}^2 has no torsion.

$$(3) \quad [\nabla_j, \nabla_k] = -4i \operatorname{Im}(\overline{b_j} b_k) \quad \begin{array}{l} j = 0, 1, \dots, n \\ k = 1, \dots, n, \end{array}$$

where

$$b_j = \frac{\partial s}{\partial x_j} / (1 + |s|^2).$$

This is either a computation, or follows from the fact that the curvature of $\{\nabla_j\}$ is the pull-back of constant curvature on \mathbb{S}^2 by the map s . The appearance of $(1 + |s|^2)$ is due to the fact that the coordinates are not (and cannot be) normalized. Note that the consistency conditions (2), (3) above include the $t = x^0$ direction.

We need the existence of a few derivatives on the map s to classically prove the following.

Theorem 1. *Let $s : \mathbb{R}^1 \times \mathbb{R}^n \rightarrow \mathbb{S}^2 = \mathbb{C}^1 \cup \{\infty\}$ be a Schrödinger map of finite energy which is asymptotic to $0 \in \mathbb{C}^1$ at spatial infinity (which can be assumed by rotation). Let*

$$\begin{aligned} u_j &= (1 + |s|^2)^{-1} e^{i\psi} \frac{\partial s}{\partial x_j} \\ D_j &= (1 + |s|^2) e^{i\psi} \circ \nabla_j \circ (1 + |s|^2) e^{-i\psi} = \\ &= \frac{\partial}{\partial x_j} + i a_j. \end{aligned}$$

Then for each t there exists a unique choice of ψ such that

$$(4) \quad \operatorname{div} a = 0; \quad a \sim 0 \quad \text{at infinity}$$

$$(5) \quad u_0 = i \sum_j D_j u_j$$

$$(6) \quad D_j u_k = D_k u_j \quad \begin{matrix} j = 0, 1, \dots, n \\ k = 1, \dots, n, \end{matrix}$$

$$(7) \quad [D_j, D_k] = i \left(\frac{\partial a_k}{\partial x_j} - \frac{\partial a_j}{\partial x_k} \right) = \\ = -4i \operatorname{Im}(u_j \overline{u_k}) \quad \begin{matrix} j = 0, 1, \dots, n \\ k = 1, \dots, n, \end{matrix}.$$

Proof. Note that (6) and (7) are gauge invariant equations. The transformation $\nabla_j \rightarrow D_j$ and $\frac{\partial s}{\partial x_j} \rightarrow u_j$ are the same gauge change. The choice of ψ is possible because

$$a_j = 2 \operatorname{Im} \left(s \frac{\overline{\partial s}}{\partial x_j} / (1 + |s|^2) \right) - \frac{\partial \psi}{\partial x_j} = \\ = 2 \operatorname{Im}(\overline{b_j} s) - \frac{\partial \psi}{\partial x_j}.$$

We simply choose a Hodge gauge with

$$\sum_{j=1}^n \frac{\partial}{\partial x_j} \left(2 \operatorname{Im}(\overline{b_j} s) - \frac{\partial \psi}{\partial x_j} \right) = 0.$$

If $\psi \sim 0$ at infinity, ψ will be unique. □

Remark Similar transformation might prove beneficial in the wave map problem as well. We hope to report on that in a later paper [5]. As we will see later, these change of variables simplifies *globally* the form of the non-linearity, which is somehow dictated by geometric considerations. Similar approach was succesfully used by T. Tao in his work on wave maps [9].

We also remark that if the map s is in fact a solution in the classical Sobolev space $L_m^p \subset C^0$, then it is not difficult to show that $b \in L_{m-1}^p$. The whole point of the gauge change is that $\frac{\partial a}{\partial x_j} \in L_{m-1}^q$, where the product $b_j \overline{b_k} \in L_{m-1}^q$. So $\{a_j\}$ are slightly smoother than the $\{b_j\}$. Of course, we will ultimately be interested in the mixed $L_t^p L_x^q$ and $X_{s,b}$ norms, but we will not go into details in this paper.

It is the set of equations in Theorem 1 which can be inverted to produce the map s . However a derived subset of these equations form a well-posed nonlinear Schrödinger flow.

Theorem 2. *The following equations, which we call the “modified Schrödinger map” (MSM) follow from the Schrödinger map equation and consistency conditions*

$$\frac{\partial u_j}{\partial t} = i \Delta u_j - 2 \sum_k a_k \frac{\partial u_j}{\partial x_k} - \left(\sum_k a_k^2 \right) u_j + \\ + 2 \operatorname{Im}(\overline{u_j} u_k) u_j - i a_0 u_j \quad j = 1, \dots, n;$$

where

$$\begin{aligned}
a_k &= \sum_{l=1}^n \frac{\partial \kappa_{lk}}{\partial x_l}; \\
d \kappa &= 0; \\
\Delta \kappa_{kj} &= -4 \operatorname{Im}(u_k \overline{u_j}) \quad \begin{matrix} j = 0, 1, \dots, n \\ k = 1, \dots, n, \end{matrix}; \\
\Delta a_0 &= -4 \sum_{j=1}^n \sum_{k=1}^n \left[\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \operatorname{Re}(u_k \overline{u_j}) - \frac{1}{2} \left(\frac{\partial}{\partial x_k} \right)^2 u_j \overline{u_j} \right].
\end{aligned}$$

Proof.

$$D_0 u_j = i[D_j, D_k]u_k + i \sum_k D_k^2 u_j$$

and on the other hand

$$D_0 u_j = D_j u_0 = i D_j \sum_k (D_k u_k),$$

Here we have used $D_k u_j = D_j u_k$. In the first equation, we see the terms

$$i[D_j, D_k]u_k = -2 \operatorname{Im}(u_j \overline{u_k})u_k.$$

and since

$$\begin{aligned}
D_k^2 &= \frac{\partial^2}{\partial x_k^2} + i a_k \frac{\partial}{\partial x_k} - a_k^2, \\
\left(\sum_k D_k^2 \right) u_j &= \Delta u_j = 2i \sum_k a_k \frac{\partial u_j}{\partial x_k} - \left(\sum_k a_k^2 \right) u_j.
\end{aligned}$$

Next, we use that

$$\operatorname{div} a = \sum_j \frac{\partial a}{\partial x_j} = 0$$

and that

$$\frac{\partial a_j}{\partial x_k} - \frac{\partial a_k}{\partial x_j} = -4 \operatorname{Im}(\overline{u_j} u_k)$$

to write

$$a = \operatorname{div} \kappa = *d * \kappa$$

and

$$\Delta \kappa = -4 \operatorname{Im}(\overline{u_j} u_k).$$

Now

$$\begin{aligned}
d a_0 - \frac{\partial}{\partial t} a &= 4 \operatorname{Im}(\overline{u_k} u_0) \\
d * a &= 0 \\
\Delta a_0 &= -4 \sum_j \frac{\partial}{\partial x_j} (\operatorname{Im} \overline{u_j} u_0) \\
u_0 &= i \sum_j D_j u_j.
\end{aligned}$$

Then

$$\begin{aligned}
\operatorname{Im}(\overline{u_k} u_0) &= - \sum_j \operatorname{Re}(\overline{u_k} (D_j u_j)) \\
&= - \sum_j \frac{\partial}{\partial x_j} \operatorname{Re}(\overline{u_k} u_j) - \operatorname{Re}(\overline{D_j u_k} u_j) \\
&= - \sum_j \frac{\partial}{\partial x_j} \operatorname{Re}(\overline{u_k} u_j) - \operatorname{Re}(\overline{D_k u_j} u_j) \\
&= \sum_j \frac{\partial}{\partial x_j} \operatorname{Re}(\overline{u_k} u_j) - \frac{1}{2} \frac{\partial}{\partial x_k} (|u_j|^2).
\end{aligned}$$

Hence,

$$\Delta a_0 = 4 \sum_j \frac{\partial}{\partial x_j} \operatorname{Re}(\overline{u_k} u_j) - \frac{1}{2} \frac{\partial}{\partial x_k} (|u_j|^2).$$

□

The modified Schrödinger map equation (MSM) is the $j = 1, \dots, n$ flows for u_j and the nonlinear operators defining the a_j 's. In this paper we prove this equation is locally well-posed when the data is in H^ϵ , $\epsilon > 0$ for $n = 2$.

It is not possible to go back directly from solutions of the MSM system to Schrödinger maps. In fact even in the one-dimensional case Chang-Shatah-Uhlenbeck ([3]) do not attempt this. In that case we have $a_1 = 0$, $a_0 = -2u_1 \overline{u_1}$. However, we sketch here a method of proving local well posedness for the Schrödinger map for data in $H^{1+\epsilon}$.

We assume that it should be possible to prove local well-posedness for data in H^k for large k for the map directly. Such solutions transform over to solutions of the complete (overdetermined) system. Our regularity results of Section 9 show that the time of existence depends only on $\|u_0\|_{H^\epsilon}$ or the $H^{1+\epsilon}$ norm of the initial data for s . Moreover, we have estimates on the differences. So given an initial data q in $H^{1+\epsilon}$, we approximate by smooth $q^\alpha \in H^k$, whose solutions u^α satisfy the full set of equations and consistency conditions. The solution produced by the well-posedness result in Theorem 3 will be a limit of the solutions in $X_{\epsilon, 1/2+}$ and hence also satisfy the entire set of consistency conditions.

As we have remarked before, the transformation formulas between $u \in X_{\epsilon, 1/2+}$ and the map s are fairly complicated. However, we are able to prove in this fashion that the equations (4), (1), (2), (3) are well-posed for initial data $u_0 \in H^\epsilon$ via this circular route; modulo the lack of a published proof that the Schrödinger map equation is locally well-posed in H^k for large k .

3. THE MODIFIED SCHRÖDINGER MAP SYSTEM

According to our reductions in the previous section, we consider the system of coupled nonlinear Schrödinger equations in R^{2+1}

$$(8) \quad \begin{cases} \partial_t u_1 &= i\Delta u_1 + 2 \left(\frac{\partial \beta}{\partial x_1} \frac{\partial u_1}{\partial x_2} - \frac{\partial \beta}{\partial x_2} \frac{\partial u_1}{\partial x_1} \right) - i\alpha u_1 - i|\nabla \beta|^2 u_1 \pm \text{Im}(u_2 \overline{u_1}) u_2, \\ \partial_t u_2 &= i\Delta u_2 + 2 \left(\frac{\partial \beta}{\partial x_1} \frac{\partial u_2}{\partial x_2} - \frac{\partial \beta}{\partial x_2} \frac{\partial u_2}{\partial x_1} \right) - i\alpha u_2 - |\nabla \beta|^2 u_2 \pm \text{Im}(u_1 \overline{u_2}) u_1 \\ u_1(x, 0) &= u_0^1(x), \\ u_2(x, 0) &= u_0^2(x). \end{cases}$$

where

$$\begin{aligned} \Delta \beta &= \pm 2 \text{Im}(u_1 \overline{u_2}), \\ \Delta \alpha &= \pm \sum_{k,j=1}^2 2 \left(\partial_{x_k} \partial_{x_j} \text{Re}(u_k \overline{u_j}) - \partial_{x_k}^2 |u_j|^2 \right). \end{aligned}$$

Our main theorem asserts that the system (8) is locally well-posed (the spaces $X_{s,b}$ are to be defined shortly).

Theorem 3. *For every $\varepsilon > 0$ and data $u_0 \in H^{100\varepsilon}$, there exists $T = T(\|u_0\|_{H^{100\varepsilon}})$, such that the system (8) has a unique solution u satisfying*

$$u \in C([0, T], H^{100\varepsilon}) \cap X_{100\varepsilon, 1/2+\varepsilon}.$$

Moreover there exists constant C_ε , independent of u_0 such that

$$(9) \quad \|u\|_{X_{100\varepsilon, 1/2+\varepsilon}} \leq C_\varepsilon \|u_0\|_{H^{100\varepsilon}}.$$

Finally, the map $u_0 \ni H^{100\varepsilon} \rightarrow u \ni C([0, T], H^{100\varepsilon}) \cap X_{100\varepsilon, 1/2+\varepsilon}$ is Lipschitz.

Essentially, we want to prove short time existence and uniqueness for data in the Sobolev space $H^{100\varepsilon}$, provided that u_1 and u_2 are components of the solution and therefore live in the same function spaces. From now on, we will not distinguish between u_1 and u_2 as they come in our formulae, as we will only use their functional analytic properties, not the fact that they are solutions. Occasionally, we will be referring to the vector $u = (u_1, u_2)$ and the data $u_0 = (u_0^1, u_0^2)$.

Next, by the Duhamel's principle, one obtains the following equivalent integral formulation for the system

$$(10) \quad u(x, t) = e^{it\Delta} u_0 + \int_0^t e^{i(t-\tau)\Delta} F_u(\tau, \cdot) d\tau,$$

where F is the nonlinearity consisting of four terms in (8). Introduce the (Schrödinger version) of the *global* Bourgain spaces $X_{s,b}$ as the set of all functions u with

$$\int |\hat{u}(\xi, \tau)|^2 < \tau - |\xi|^2 >^{2b} < \xi >^{2s} d\xi d\tau < \infty,$$

where $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ and $\langle \tau - |\xi|^2 \rangle := (1 + |\tau - |\xi|^2|^2)^{1/2}$. We also introduce the space $X_{s,b}^-$ as

$$X_{s,b}^- := \left\{ u : \int |\hat{u}(\xi, \tau)|^2 \langle \tau + |\xi|^2 \rangle^{2b} \langle \xi \rangle^{2s} d\xi d\tau < \infty \right\}.$$

Note that the dual space to $X_{s,b}$ is $X_{-s,-b}^-$. Sometimes we will refer to s as the amount of elliptic smoothness in $X_{s,b}$ and to b as the parabolic smoothness for the corresponding space. We will also need a *local* version of the $X_{s,b}$ spaces, since our solutions are local in nature. Define

$$\|u\|_{X_{s,b}([0,T] \times \mathbb{R}^2)} = \inf \left\{ \|U\|_{X_{s,b}} : U|_{[0,T] \times \mathbb{R}^2} = u \right\}.$$

Sometimes, we will not distinguish between the local and the global spaces. Our estimates are performed on a solution cut off in time in the global space. Thus, they are in particular estimates on the solution in the local space with time interval given by the lifespan of the solution.

It is well known (see for example [7]) that the Schrödinger semigroup $e^{it\Delta}$ has certain smoothing effect on the parabolic derivatives. More precisely, let ψ be a smooth characteristic function of the interval $(-1, 1)$ and $1 - \varepsilon > b > 1/2$ for some positive $\varepsilon > 0$. Then

$$(11) \quad \left\| \psi(\delta^{-1}t) \int_0^t e^{i(t-\tau)\Delta} F(\tau, \cdot) d\tau \right\|_{X_{s,b}} \leq C_\varepsilon \delta^\varepsilon \|F\|_{X_{s,b-1+\varepsilon}}.$$

The following estimate for the growth of the free solution $e^{it\Delta}u_0$ in the Bourgain spaces is also well-known (see [7], Lemma 2.3.1).

Lemma 1. *For $s \geq 0$ and $b > 1/2$, $0 < \delta_0 \leq 1$, we have*

$$\|\psi(t/\delta_0)e^{it\Delta}u_0\|_{X_{s,b}} \lesssim \delta_0^{(1-2b)/2} \|u_0\|_{H^s}.$$

The approach for solving (8) is by the method of Pickard iterations. Therefore, to prove short-time existence (and uniqueness in certain class), one needs to show that the map

$$\Phi(v) = \psi(t/\delta_0)e^{it\Delta}u_0 + \psi(\delta^{-1}t) \int_0^t e^{i(t-\tau)\Delta} F_v(\tau, \cdot) d\tau,$$

is a contraction on a ball in a suitable Banach space. We choose $X_{100\varepsilon, 1/2+\varepsilon} \times X_{100\varepsilon, 1/2+\varepsilon}$, if our data $u_0 \in H^{100\varepsilon} \equiv X_{100\varepsilon, 0}$. Some remarks are in order.

- By scaling and dimensional analysis, it is easy to see that if $u(x, t)$ solves the initial value problem (8), then $u_\alpha(x, t) = \alpha u(\alpha x, \alpha^2 t)$ solves the same with data $\alpha u_0(\alpha x)$. Hence the system (8) is critical in $L^2(\mathbb{R}^2)$ (i.e. the critical index is $s_c = 0$). It is interesting to note that although the term $|\nabla \beta|^2 u_j$ is essentially quintic, it scales and -as we shall see- behaves like a cubic nonlinearity. Indeed general quintic nonlinearities have critical index $s_c = 1/2$ in two dimensions. However our quintic is very special and it actually has $s_c = 0$.
- The terms $i\alpha u_k$ and $\pm i \text{Im}(u_k \overline{u_j}) u_k$ have approximately the structure of a *cubic nonlinearity* $|u|^2 u$. These have been extensively studied and estimates have been obtained

in various Lebesgue and Besov spaces. Nevertheless, the methods in [4] and [8] allow us to estimate in the Schrödinger $X_{s,b}$ spaces. We will refer to these terms as F_{cubic} .

- The terms $|\nabla\beta|^2 u_k$ are essentially *quintic* in u , which are in general handled in spaces with at least *half* derivative on the data. However, as we shall see later, two of the entries in the five-linear forms come with a “missing derivatives”. This allows us to use Sobolev embedding in conjunction with our new embedding theorem for $X_{s,b}$ spaces to get the estimates with minimal smoothness assumptions. We refer to these as $F_{quintic}$. More specifically,

$$F_{quintic}(u_1, u_2, u_3, u_4, u_5) = \nabla \Delta^{-1}(u_1 \overline{u_2}) \nabla \Delta^{-1}(u_3 \overline{u_4}) u_5.$$

- For the first term, more refined analysis is needed, since it involves derivatives of the solution. On the other hand, we need to control this expression with virtually no regularity present, and to this end one needs to exploit the “null form” structure. Observe that the “null form” nonlinearity is also (anti) trilinear. We call it $F_{null}(u_1, u_2, u_3)$, where

$$F_{null}(u_1, u_2, u_3) = \frac{\partial \beta}{\partial x_1} \frac{\partial u_3}{\partial x_2} - \frac{\partial \beta}{\partial x_2} \frac{\partial u_3}{\partial x_1}.$$

Since we do not distinguish between u_1 and u_2 in our estimates (we simply assume that they are in $X_{100\varepsilon, 1/2+\varepsilon}$), it will suffice to prove

$$(12) \quad \|F_{cubic}(u_1, u_2, u_3)\|_{X_{100\varepsilon, -1/2+2\varepsilon}} \lesssim \|u_1\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|u_2\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|u_3\|_{X_{100\varepsilon, 1/2+\varepsilon}}.$$

$$(13) \quad \|F_{null}(u_1, u_2, u_3)\|_{X_{100\varepsilon, -1/2+2\varepsilon}} \lesssim \|u_1\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|u_2\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|u_3\|_{X_{100\varepsilon, 1/2+\varepsilon}}.$$

and

$$(14) \quad \|F_{quintic}(u_1, u_2, u_3, u_4, u_5)\|_{X_{100\varepsilon, 1/2+\varepsilon}} \lesssim \prod_{j=1}^5 \|u_j\|_{X_{100\varepsilon, 1/2+\varepsilon}}$$

Then (11), (12), (13), (14) and Lemma 1 imply that

$$\Phi : B_{R_0}(X_{100\varepsilon, 1/2+\varepsilon}) \times B_{R_0}(X_{100\varepsilon, 1/2+\varepsilon}) \rightarrow B_{R_0}(X_{100\varepsilon, 1/2+\varepsilon}) \times B_{R_0}(X_{100\varepsilon, 1/2+\varepsilon})$$

is a contraction mapping for suitably chosen R_0 and δ .

Indeed, our estimates for the nonlinearities will suffice to show that for $v = (v_1, v_2)$

$$\begin{aligned} \|\Phi(v)\|_{X_{100\varepsilon, 1/2+\varepsilon} \times X_{100\varepsilon, 1/2+\varepsilon}} &\lesssim \|u_0\|_{H^{100\varepsilon}} + \delta^\varepsilon \prod_{j=1}^3 \|v_j\|_{X_{100\varepsilon, 1/2+\varepsilon}} + \\ &+ \delta^\varepsilon \prod_{j=1}^5 \|v_j\|_{X_{100\varepsilon, 1/2+\varepsilon}}, \end{aligned}$$

where $\delta_0 = 1$ from Lemma 1 and $v_j = v_1$ or v_2 for $j = 3, 4, 5$. Hence if $R_0 \sim \|u_0\|_{H^{100\varepsilon}}$ and $0 < \delta \leq \delta_0$ satisfies

$$\delta < \min \left(\frac{1}{\|u_0\|_{H^{100\varepsilon}}^{2/\varepsilon}}, \frac{1}{\|u_0\|_{H^{100\varepsilon}}^{4/\varepsilon}} \right),$$

we have that

$$\Phi : B_{R_0} \times B_{R_0} \rightarrow B_{R_0} \times B_{R_0}$$

and that there exists $0 < C_0 < 1$ such that

$$\|\Phi(v) - \Phi(\tilde{v})\|_{X_{100\varepsilon, 1/2+\varepsilon} \times X_{100\varepsilon, 1/2+\varepsilon}} \leq C_0 \|v - \tilde{v}\|_{X_{100\varepsilon, 1/2+\varepsilon} \times X_{100\varepsilon, 1/2+\varepsilon}}.$$

In particular we have estimates on the differences. The above is also enough to establish the Lipschitz bounds that were claimed in Theorem 3.

The following Lemma yields the estimates needed to handle the cubic-like nonlinearities.

Lemma 2.

$$(15) \quad \|u_1 \overline{u_2} u_3\|_{X_{s,-1/2+2\varepsilon}} \lesssim \|u_1\|_{X_{s,1/2+\varepsilon}} \|u_2\|_{X_{s,1/2+\varepsilon}} \|u_3\|_{X_{s,1/2+\varepsilon}}$$

$$(16) \quad \|u_1 \overline{u_2} \overline{u_3}\|_{X_{s,-1/2+2\varepsilon}} \lesssim \|u_1\|_{X_{s,1/2+\varepsilon}} \|u_2\|_{X_{s,1/2+\varepsilon}} \|u_3\|_{X_{s,1/2+\varepsilon}}$$

$$(17) \quad \|u_1 u_2 u_3\|_{X_{s,-1/2+2\varepsilon}} \lesssim \|u_1\|_{X_{s,1/2+\varepsilon}} \|u_2\|_{X_{s,1/2+\varepsilon}} \|u_3\|_{X_{s,1/2+\varepsilon}}$$

provided $s > 5\varepsilon$.

We will prove Lemma 2, together with (12) for the cubic nonlinearities in Section 5. The mixed space-time Lebesgue spaces are defined as the set of all functions u with

$$\|u\|_{L_t^p L_x^q} = \left(\int \left(\int |u(x,t)|^q dx \right)^{p/q} dt \right)^{1/p}.$$

The next lemma supplies an embedding theorem for $X_{s,b}$ spaces into a mixed norm Lebesgue spaces¹

Lemma 3. For $1 \leq p \leq 2$,

$$(18) \quad X_{0,1/2+} \hookrightarrow L_t^{2p'} L_x^{2p}(\mathbb{R}^2 \times \mathbb{R}^1)$$

More generally, we have the bilinear estimates

$$(19) \quad \|uv\|_{L_t^{p'} L_x^p} \lesssim \|u\|_{X_{0,1/2+\varepsilon}} \|v\|_{X_{0,1/2+\varepsilon}}.$$

$$(20) \quad \|u\overline{v}\|_{L_t^{p'} L_x^p} \lesssim \|u\|_{X_{0+,1/2+\varepsilon}} \|v\|_{X_{0+,1/2+\varepsilon}}$$

We prove Lemma 3 as well as the estimates required for the quintic nonlinearities in Section 6.

Next, note that the “null form” nonlinearity F_{null} has two components (one for each equation in the system), but both components look identical except for the dependence upon u_1 and u_2 , which is irrelevant in our argument. Thus, it suffices to consider only one component. We test F_{null} against a function $W \in X_{-100\varepsilon,1/2-2\varepsilon}^-$ to get the (anti) multilinear form

$$M(u_3, u_1, u_2, W) = \int \left(\frac{\partial \beta}{\partial x_1} \frac{\partial u_3}{\partial x_2} - \frac{\partial \beta}{\partial x_2} \frac{\partial u_3}{\partial x_1} \right) \overline{W} dx dt.$$

Hence, the following theorem takes care of the null-form nonlinearity.

Theorem 4.

$$(21) \quad |M(u_1, u_2, u_3, W)| \lesssim \|u_3\|_{X_{100\varepsilon,1/2+\varepsilon}} \|u_1\|_{X_{100\varepsilon,1/2+\varepsilon}} \|u_2\|_{X_{100\varepsilon,1/2+\varepsilon}} \|\overline{W}\|_{X_{-100\varepsilon,1/2-2\varepsilon}},$$

where $\Delta\beta = \pm 2Im(u_1 \overline{u_2})$.

¹The embedding (18) actually holds in more generality into the $L_t^q L_x^r$ spaces with $2/q + n/r = n/2$ and n the space dimension. The proof of it relies on the Strichartz inequalities and it was pointed out to us by T. Tao after we had derived the embedding -in our restricted range- as a consequence of our inequalities (19) and (20). Since we actually need (19) and (20) per se to treat the quintic nonlinearity we prefer to keep the statement of such result in the fashion of Lemma 2 above.

We make some reductions for the proof of Theorem 4. First, we will assume that all four functions u_3, u_1, u_2, W are test functions with norm one in the corresponding spaces. Standard approximation techniques will then yield the general result.

Next, we perform integration by parts in the definition of M to take the derivatives off β . The special cancellation properties of the expression and the lack of boundary terms allow us to rewrite M as

$$M(u_3, u_1, u_2, W) = \int \beta \left(\frac{\partial \bar{W}}{\partial x_2} \frac{\partial u_3}{\partial x_1} - \frac{\partial \bar{W}}{\partial x_1} \frac{\partial u_3}{\partial x_2} \right).$$

Since $\Delta\beta = \pm 2Im(u_1 \bar{u}_2) = \pm(u_1 \bar{u}_2 - u_2 \bar{u}_1)$, two similar terms arise. By slightly abusing our notations, we will call one of them (say the one that corresponds to $u_1 \bar{u}_2$) M . We have

$$(22) \quad M(u_3, u_1, u_2, W) = \int u_1 \bar{u}_2 G,$$

where

$$G = \Delta^{-1} \left(\frac{\partial \bar{W}}{\partial x_2} \frac{\partial u_3}{\partial x_1} - \frac{\partial \bar{W}}{\partial x_1} \frac{\partial u_3}{\partial x_2} \right).$$

Parseval's identity, together with $\widehat{u}_2(\xi, \tau) = \widehat{u}_2(-\xi, -\tau)$ imply that

$$(23) \quad M(u_3, u_1, u_2, W) = \int u_1 G \bar{u}_2 = \int \widehat{u_1 G}(\xi, \tau) \widehat{u}_2(-\xi, -\tau) d\xi d\tau.$$

Since the complex conjugation is an isometry of $X_{s,b}$ onto $X_{s,b}^-$, we can write

$$\widehat{u}_2(-\xi, -\tau) = \frac{h_2(-\xi, -\tau)}{< \tau - |\xi|^2 >^{1/2+\varepsilon}} < \xi >^{-100\varepsilon},$$

for some $h_2 \in L^2$.

Similarly by using the properties of $X_{s,b}$ spaces, we express $u_1(u_2)$ via its L^2 representative h_1 (h_2 respectively) and some weights dictated by the particular space. We get,

$$(24) \quad \begin{aligned} M(u_3, u_1, u_2, W) &= \int \frac{h_1(\xi - \eta, \tau - \mu) < \xi - \eta >^{-100\varepsilon}}{< \tau - \mu - |\xi - \eta|^2 >^{1/2+\varepsilon}} \widehat{G}(\eta, \mu) \times \\ &\times \frac{h_2(-\xi, -\tau) < \xi >^{-100\varepsilon}}{< \tau - |\xi|^2 >^{1/2+\varepsilon}} d\xi d\eta d\tau d\mu, \end{aligned}$$

where

$$(25) \quad \begin{aligned} \widehat{G}(\eta, \mu) &= \frac{1}{|\eta|^2} \int (\eta_1 z_2 - \eta_2 z_1) \widehat{W}(\eta - z, \mu - s) \widehat{u}_3(z, s) dz ds = \\ &= \frac{1}{|\eta|^2} \int \langle \eta, z^\perp \rangle \widehat{W}(\eta - z, \mu - s) \widehat{u}_3(z, s) dz ds. \end{aligned}$$

We will need the representation of M as a quadrilinear form applied to four L^2 functions. Meanwhile, we change variables $\eta \rightarrow \eta + z$ and $z \rightarrow -z$ to obtain

$$(26) \quad \begin{aligned} \Lambda(h_1, h_2, f, g) &= \int \frac{h_1(\xi - \eta + z, \tau - \mu) < \xi - \eta + z >^{-100\varepsilon}}{< \tau - \mu - |\xi - \eta + z|^2 >^{1/2+\varepsilon}} \frac{h_2(-\xi, -\tau) < \xi >^{-100\varepsilon}}{< \tau - |\xi|^2 >^{1/2+\varepsilon}} \\ &\times \frac{\langle \eta, z^\perp \rangle}{|\eta - z|^2} \frac{f(\eta, \mu - s) < \eta >^{100\varepsilon}}{< \mu - s - |\eta|^2 >^{1/2-2\varepsilon}} \frac{g(-z, s) < z >^{-100\varepsilon}}{< s - |z|^2 >^{1/2+\varepsilon}} d\xi d\eta dz d\tau d\mu ds. \end{aligned}$$

We break up the η and z integration in the definition of Λ to obtain

$$\Lambda(h_1, h_2, f, g) = \int_{|z| \leq |\eta|/2} \dots + \int_{|z| \sim |\eta|} \dots = \Lambda_{offdiag} + \Lambda_{diag}$$

We will estimate $\Lambda_{offdiag}$ in Section 7 and Λ_{diag} in Section 8.

4. SOME REMARKS REGARDING MULTILINEAR FORMS

In this section, we follow [8] to introduce somewhat more general framework for the multilinear forms that we have to deal with.

For an integer k and abelian group Z , define the hyperplane

$$\Gamma_k(Z) = \{(\xi_1, \dots, \xi_k) \in Z^k : \xi_1 + \dots + \xi_k = 0\}.$$

A $[k, Z]$ multiplier is a function $m : \Gamma_k(Z) \rightarrow \mathcal{C}$, so that there exists a constant C , such that the inequality

$$\left| \int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^k f_j(\xi_j) \right| \leq C \prod_{j=1}^k \|f_j\|_{L^2(Z)},$$

holds for all $f_1, \dots, f_k \in L^2(Z)$. The best constant C with the above property is naturally called a multiplier norm for m and is usually denoted $\|m\|_{[k, Z]}$. We will also need the notation $\Gamma_k(Z, \xi_j = \eta_j) := \Gamma_k(Z) \cap \{\xi_j = \eta_j\}$ for a fixed η_j .

The Cauchy-Schwarz inequality gives the following very useful lemma. (cf. [8], Lemma 3.9)

Lemma 4. *If m is a $[k, Z]$ multiplier, then*

$$\|m\|_{[k, Z]} \leq \sup_{\eta_j \in Z} \left(\int_{\Gamma_k(Z, \xi_j = \eta_j)} |m(\xi)|^2 \right)^{1/2}.$$

We will also need the following corollary for the cases $k = 3, 4$ (cf. [8], Corollary 3.10).

Corollary 1. *For any subsets A, B, C of Z , we have*

$$(27) \quad \|\chi_A(\xi_1)\chi_B(\xi_2)\|_{[3, Z]} \leq |\{\xi_1 \in A : \xi - \xi_1 \in B\}|^{1/2},$$

$$(28) \quad \|\chi_A(\xi_1)\chi_B(\xi_2)\chi_C(\xi_3)\|_{[4, Z]} \leq |\{(\xi_1, \xi_2) \in A \times B : \xi - \xi_1 - \xi_2 \in C\}|^{1/2}$$

for some $\xi \in Z$.

Proof. The proofs of (27) and (28) are similar, so we show only (28). By Lemma 4,

$$\begin{aligned} \|\chi_A(\xi_1)\chi_B(\xi_2)\chi_C(\xi_3)\|_{[4, Z]} &\leq \sup_{\eta_4 \in Z} \left(\int_{\Gamma_k(Z, \xi_4 = \eta_4)} |\chi_A(\xi_1)|^2 |\chi_B(\xi_2)|^2 |\chi_C(\xi_3)|^2 \right)^{1/2} \lesssim \\ &\lesssim \| |\chi_A \otimes \chi_B| * |\chi_C| \|_{L^\infty(Z)}^{1/2} = |\{(\xi_1, \xi_2) \in A \times B : \xi - \xi_1 - \xi_2 \in C\}|^{1/2}. \end{aligned}$$

□

The other important technical tool that is crucial for us will be a form of the Schur's test and the box localization method (cf. Lemma 3.11 and Corollary 3.13 in [8]). We define the j 'th support of m to be the set

$$\text{supp}_j(m) = \{\eta_j \in Z : \Gamma_k(Z, \xi_j = \eta_j) \cap \text{supp}(m) \neq \emptyset\}.$$

In particular, if $m(\xi_1, \dots, \xi_s) = \prod_{j=1}^s m_j(\xi_j)$, we have $\text{supp}_j(m) = \text{supp}(m_j)$.

Lemma 5. *Let R be a rectangular box in Z . Suppose also that $\text{supp}_j(m)$ is contained in a $R + \eta_j$ for some $\eta_j \in Z$ and $1 \leq j \leq k-2$. Then*

$$\|m\|_{[k,Z]} \sim \sup_{\eta_{k-1}, \eta_k \in Z} \|m \chi_{R+\eta_{k-1}}(\xi_{k-1}) \chi_{R+\eta_k}(\xi_k)\|_{[k,Z]}.$$

In particular, the box localization principle states, that if we have a multiplier in which all but two of the variables are restricted to sets of certain diameter, we can restrict (at the expense of a bigger constant) the remaining two variables to sets with the same diameter.

Next, consider a fixed smooth function ψ on \mathbb{R}^1 supported around 1. Introduce also a cutoff ψ_0 on \mathbb{R}^1 supported around 0 with

$$\psi_0(\cdot) + \sum_{R \geq 1} \psi(R^{-1}(\cdot)) \equiv 1, \quad \text{on } \mathbb{R}^1.$$

Let $\widehat{u_R} = \widehat{S_R u} = \psi(R^{-1}|\cdot|)\widehat{u}(\cdot, t)$ be the standard Littlewood-Paley operator *in space* at frequency R applied to the function $u(\cdot, t)$. Sometimes we will slightly abuse notations by using f_R to denote the restriction of the function $f(\xi)$ to the annulus $\{\xi : |\xi| \sim R\}$. We also introduce the following notation: for every sequence of real numbers L_1, \dots, L_n , the sequence L_1^*, \dots, L_n^* will denote the permutation in *increasing* order of the original sequence.

We need the following lemmas, which are adaptation of estimates (80) and (86) – (89) in [8] (cf. Proposition 11.1 and Proposition 11.2 in [8]).

Lemma 6. *For the $[3, R^{d+1}]$ multiplier $m_1(\xi, \tau) = \prod_{i=1}^3 \chi_{\tau_i - |\xi_i|^2 \sim L_i} \chi_{|\xi_i| \sim R_i}$, we have the estimate*

$$\|m_1\|_{[3, R^{d+1}]} \lesssim L_1^{*1/2} R_3^{*-1/2} R_1^{*(d-1)/2} \min(R_1^* R_3^*, L_2^*)^{1/2}.$$

Lemma 7. *For $\xi_0 \in \mathbb{R}^d$ and the multiplier*

$$m_2(\xi, \tau) = \chi_{\tau_1 - |\xi_1|^2 \sim L_1} \chi_{\tau_2 + |\xi_2|^2 \sim L_2} \chi_{|\xi_1 - \xi_0| \leq r, |\xi_2 + \xi_0| \leq r, |\xi_3| \leq r}$$

we have

- *If $|\xi_0| \lesssim r$, then*

$$(29) \quad \|m_2\|_{[3, R^{d+1}]} \lesssim \min(L_1, L_2)^{1/2} r^{(d-2)/2} \min(r^2, \max(L_1, L_2))^{1/2}.$$

- *If $|\xi_0| \gg r$ and $H = ||\xi_1|^2 + |\xi_3|^2 - |\xi_2|^2| \sim \tau_3 - |\xi_3|^2$, $H \lesssim |\xi_0| r$ then*

$$(30) \quad \|m_2\|_{[3, R^{d+1}]} \lesssim \min(L_1, L_2)^{1/2} \frac{r^{(d-1)/2}}{|\xi_0|^{1/2}} \min\left(H, \frac{H}{r^2} \max(L_1, L_2)\right)^{1/2}.$$

- *If $|\xi_0| \gg r$ and $H \approx \tau_3 - |\xi_3|^2$, $H \lesssim |\xi_0| r$ then*

$$(31) \quad \|m_2\|_{[3, R^{d+1}]} \lesssim \min(L_1, L_2)^{1/2} \frac{r^{(d-1)/2}}{|\xi_0|^{1/2}} \min(|\xi_0| r, \max(L_1, L_2))^{1/2}.$$

Proof. The proof is a reprise of the argument behind Proposition 11.2 in [8], so we will just indicate the main points.

- If $|\xi_0| \lesssim r$, we imagine that we have the additional restriction $\tau_3 - |\xi_3|^2 \sim L$ in the multiplier m_2 , which will be artificial and it will not play any role. Compute $|H| = ||\xi_1|^2 + |\xi_3|^2 - |\xi_2|^2| \lesssim r^2$. Thus, we might be in any of the cases (86), (88), (89) in [8], but in all of them, we get

$$\|m_2\|_{[3, R^{d+1}]} \lesssim \min(L_1, L_2)^{1/2} r^{(d-2)/2} \min(r^2, \max(L_1, L_2))^{1/2}.$$

- If $|\xi_0| \gg r$, we will impose again the additional artificial restriction $\tau_3 - |\xi_3|^2 \sim L$. Then $|H| = ||\xi_1|^2 + |\xi_3|^2 - |\xi_2|^2| \lesssim |\xi_0|r$ and the rest follows from Proposition 11.2 in [8].

□

Remark Geometric considerations indicate that the restriction $\tau_k \pm |\xi_k|^2 \sim L_k \sim L_3^*$ is very weak or redundant altogether. Actually, in the proof of Lemma 6 and Lemma 7 (see the discussion and the reductions in [8], estimates (33) – (40)), one always estimates by the norm of the multiplier, where the restriction $\tau_k \pm |\xi_k|^2 \sim L_k \sim L_3^*$ is not present. Later on, when we need to break up the integrals in the multilinear forms relative to the size of the weights $\tau_i \pm |\xi_i|^2$, we will implicitly use the fact that the restriction $\tau_k \pm |\xi_k|^2 \sim L_k$ ($L_k \sim L_3^*$) does not appear.

The lemma below appears in [4] and essentially follows by combining various cases in Proposition 11.2 of [8]. We state it separately, since it will be used in this form in the sequel.

Lemma 8. (*Estimate 2.20 in [4]*) *For the trilinear form*

$$C(f, g, h) = \int_{\Gamma_3(R^{2+1})} f_R(\xi_1, \tau_1) \frac{g_M(\xi_2, \tau_2)}{< \tau_1 - |\xi_1|^2 >^{1/2+\varepsilon}} \frac{h_N(\xi_3, \tau_3)}{< \tau_2 + |\xi_2|^2 >^{1/2+\varepsilon}}$$

there is the estimate

$$(32) \quad |C(f, g, h)| \lesssim \left(\frac{\min(M, N)}{\max(M, N)} \right)^{1/2} \|f_R\|_{L^2(\mathbb{R}^3)} \|g_M\|_{L^2(\mathbb{R}^3)} \|h_N\|_{L^2(\mathbb{R}^3)}.$$

As a corollary, one has the estimate for products

$$(33) \quad \|S_R((u_1)_M(\overline{u_2})_N)\|_{L^2(\mathbb{R}^3)} \lesssim \left(\frac{\min(M, N)}{\max(M, N)} \right)^{1/2} \max(M, N)^{-s} \|u_1\|_{X_{s, 1/2+\varepsilon}} \|u_2\|_{X_{s, 1/2+\varepsilon}} \\ \lesssim \left(\frac{\min(M, N)}{\max(M, N)} \right)^{1/2} R^{-s} \|u_1\|_{X_{s, 1/2+\varepsilon}} \|u_2\|_{X_{s, 1/2+\varepsilon}}.$$

We have used $\max(M, N) \gtrsim R$, since otherwise $\text{supp } \mathcal{F}((u_1)_M(\overline{u_2})_N) \subset \{\xi : |\xi| \ll R\}$ and $S_R((u_1)_M(\overline{u_2})_N) = 0$.

5. CUBIC NONLINEARITIES

We start off with a proposition, that is essentially equivalent to Lemma 2. Later on, we will also use it in the *off-diagonal* considerations for the “null form” nonlinearity.

Proposition 1. *Let m be a bounded function. Then for every $\kappa : 1/2 > \kappa > 5\varepsilon$, the quadrilinear form*

$$\begin{aligned} H(f, g, h, w) &= \int_{\substack{\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0, \\ \tau_1 + \tau_2 + \tau_3 + \tau_4 = 0}} m(\xi_i) \frac{f(\xi_1, \tau_1) < \xi_1 >^{-\kappa}}{< \tau_1 - |\xi_1|^2 >^{1/2+\varepsilon}} \frac{g(\xi_2, \tau_2) < \xi_2 >^{-\kappa}}{< \tau_2 \pm |\xi_2|^2 >^{1/2+\varepsilon}} \\ &\times \frac{h(\xi_3, \tau_3) < \xi_3 >^{\kappa}}{< \tau_3 - |\xi_3|^2 >^{1/2-2\varepsilon}} \frac{w(\xi_4, \tau_4) < \xi_4 >^{-\kappa}}{< \tau_4 \pm |\xi_4|^2 >^{1/2+\varepsilon}} d\xi_1 \dots d\xi_4 d\tau_1 \dots d\tau_4. \end{aligned}$$

satisfies

$$|H(f, g, h, w)| \lesssim \|m\|_{\infty} \|f\|_2 \|g\|_2 \|h\|_2 \|w\|_2.$$

Remark We will need to have a nontrivial m to cover some cases where we have a *zero* order pseudodifferential operators acting on some of the entries.

Proof. Observe that one can assume $m = 1$ without loss of generality since all the proofs proceed by putting absolute values inside the integrals anyway.

For the case of weight $\tau_4 - |\xi_4|^2$ and $\tau_2 + |\xi_2|^2$ write

$$\begin{aligned} \tilde{f}(\xi, \tau) &= \frac{f(\xi, \tau) < \xi >^{-\kappa}}{< \tau - |\xi|^2 >^{1/2+\varepsilon}} \\ \tilde{g}(\xi, \tau) &= \frac{g(\xi, \tau) < \xi >^{-\kappa}}{< \tau + |\xi|^2 >^{1/2+\varepsilon}} \end{aligned}$$

By taking a dyadic decomposition on the space frequency, we have

$$H = \sum_R \int S_R(\tilde{f}\tilde{g}) S_R(\tilde{G}) = \sum_{M,N,R} \int S_R(\tilde{f}_M \tilde{g}_N) \tilde{G}_R d\eta d\mu,$$

where

$$\widehat{\tilde{G}}(\eta, \mu) = \int \frac{h(\eta - z, \mu - s) < \eta - z >^{\kappa}}{< \mu - s - |\eta - z|^2 >^{1/2-2\varepsilon}} \frac{w(z, s)}{< z >^{\kappa} < s - |z|^2 >^{1/2+\varepsilon}} dz ds.$$

An application of Cauchy-Schwarz and (33) from Lemma 8 yield

$$|H| \leq \sum_{M,N,R} \left\| S_R(\tilde{f}_M \tilde{g}_N) \right\|_{L^2} \left\| \tilde{G}_R \right\|_{L^2} \lesssim \sum_R R^{-\kappa} \|f\|_2 \|g\|_2 \left\| \tilde{G}_R \right\|_{L^2}.$$

To complete the desired estimate it will suffice to show that

$$(34) \quad \sum_{R>1} R^{-\kappa} \left\| \tilde{G}_R \right\|_{L^2} \lesssim 1.$$

We test $\widehat{\tilde{G}}_R$ against an unimodular L^2 function v to get the trilinear form S

$$\begin{aligned} (35) \quad S(h, w, v) &:= \langle \widehat{\tilde{G}}_R, v \rangle = \sum_{r,R} \int \frac{h_R(\eta - z, \mu - s) < \eta - z >^{\kappa}}{< \mu - s - |\eta - z|^2 >^{1/2-2\varepsilon}} \times \\ &\times \frac{w_r(z, s)}{< z >^{\kappa} < s - |z|^2 >^{1/2+\varepsilon}} v_{\max(r,R)}(\eta, \mu) d\eta dz d\mu ds \end{aligned}$$

applied to the L^2 functions h, w, v . We estimate away the elliptic weights $\langle \eta - z \rangle^\kappa \langle z \rangle^{-\kappa}$ to get

$$\begin{aligned} S(h, w, v) &\lesssim = \sum_{r \leq R} \left(\frac{R}{r} \right)^\kappa \int \frac{h_R(\eta - z, \mu - s)}{\langle \mu - s - |\eta - z|^2 \rangle^{1/2-2\varepsilon}} \times \\ &\quad \times \frac{w_r(z, s)}{\langle s - |z|^2 \rangle^{1/2+\varepsilon}} v_R(\eta, \mu) d\eta dz d\mu ds. \end{aligned}$$

for positive h, w, v . Let $L_1 = \mu - s - |\eta - z|^2$, $L_2 = s - |z|^2$ and $L_3 = \mu - |\eta|^2$. An application of Lemma 6 yields

$$\begin{aligned} |S(h, w, v)| &\lesssim \sum_{r \leq R} \sum_{L_1^*, L_2^*} \frac{L_1^{*1/2} \min(L_2^*, Rr)^{1/2}}{\langle L_1 \rangle^{1/2-2\varepsilon} \langle L_2 \rangle^{1/2+\varepsilon}} \|h_R\| \|w_r\| \|v_R\| \left(\frac{r}{R} \right)^{1/2-\kappa} \lesssim \\ &\lesssim \sum_{r \leq R} \sum_{L_2^* < Rr} L_2^{*2\varepsilon} \|h_R\| \|w_r\| \|v_R\| \left(\frac{r}{R} \right)^{1/2-\kappa} + \\ &+ \sum_{r \leq R} \sum_{L_2^* > Rr} \frac{(Rr)^{1/2}}{L_2^{*1/2-2\varepsilon}} \|h_R\| \|w_r\| \|v_R\| \left(\frac{r}{R} \right)^{1/2-\kappa} \lesssim R^{4\varepsilon} \|h\| \|w\| \|v\|. \end{aligned}$$

Thus $\|\tilde{G}_R\|_{L^2} \lesssim R^{4\varepsilon}$ and therefore (34) holds.

For the case of weight $\tau_4 + |\xi|^2$, $\tau_2 + |\xi_2|^2$ we need to consider different pairing of our functions: f, h versus g, w rather than f, g versus h, w as we have just done. One obtains two trilinear forms where the weights have *the same* signs and we apply Lemma 6 to each one of them. Then we perform similar and in fact simpler argument to the one presented above. In the case $\tau_2 - |\xi_2|^2$, $\tau_4 - |\xi_4|^2$ we once again pair functions with *same signs* weights and use Lemma 6. Finally, in the case $\tau_4 + |\xi_4|^2$, $\tau_2 - |\xi_2|^2$ the argument is identical to the one presented above. We omit the details. \square

As we have already mentioned, Proposition 1 implies Lemma 2. We will show now (12) for all cubic-like nonlinearities. We have from the defining relations for α

$$\alpha_j = i \sum_{k=1}^2 R_j R_k \operatorname{Re}(u_k \overline{u_j}) - i|u_1|^2 - i|u_2|^2,$$

where R_j is the Riesz transform in the j th variable. Since the multiplier corresponding to R_j is $\xi_j/|\xi|$, (12) for the nonlinearity αu_j reduces to Proposition 1 with a suitable choice of m .

6. QUINTIC NONLINEARITIES

In this section, we will estimate the terms with a “quintic” nonlinearities. As it was mentioned earlier, quintic nonlinearities are difficult to control in a space with less than a half derivative. We have however a very special form of the nonlinearity, which makes it tractable. A good model expression of what we are dealing with is

$$P_{-1}(u_1 \overline{u_2}) P_{-1}(u_3 \overline{u_4}) u_5,$$

where P_{-1} will be a smoothing pseudodifferential operator of order -1 . At the first step, we use the cubic estimates outlined in Section 5 and then we exploit the “smoothing” provided by P_{-1} via Sobolev embedding. To accomplish this program, we will need to pass from $X_{s,b}$ to a mixed norm Lebesgue spaces and back. It is also a question of independent interest to study the relation between the $X_{s,b}$ spaces and the mixed norm Lebesgue spaces. Actually, the reader is probably aware that most of the current existence results are in fact proved via an appropriate contraction mapping argument in mixed Lebesgue spaces of various sorts. We also point out that the case $p = 2$ in Lemma 3 is the well-known Bourgain’s lemma ([1], Corollary 3.39)

Proof. (Lemma 3). First, we note that the bilinear estimate (19) implies the embedding (18), if one takes $u = v$. As it has been already noted, the endpoint $p = 2$ is contained in Corollary 3.39 of [1], but can also be obtained from Lemma 6. For the bilinear estimate (20), one uses as an endpoint L^2 result (32). Note that one needs a little bit of extra regularity in u, v ($X_{0+,1/2+}$) in order to be able to sum (20) in R .

By complex interpolation, it will suffice to show the other endpoint $p = 1$. The proof for both (19) and (20) is the same for $p = 1$, so we concentrate on (19). Since $\|uv\|_{L_x^1} \leq \|u\|_{L_x^2} \|v\|_{L_x^2}$, we reduce it to showing

$$\sup_t \|u\|_{L^2} \lesssim \|u\|_{X_{0,1/2+}}.$$

Consider a function $f \in L^2(\mathbb{R}^2 \times \mathbb{R}^1)$, such that

$$\widehat{u}(\xi, \tau) = f(\xi, \tau) < \tau - |\xi|^2 >^{-1/2-\varepsilon}.$$

We have

$$\begin{aligned} \sup_t \|u\|_{L_x^2} &= \sup_t \left\| \int \frac{f(\xi, \tau)}{< \tau - |\xi|^2 >^{1/2+\varepsilon}} e^{i\tau t} d\tau \right\|_{L_x^2} \lesssim \\ &\lesssim \left\| \|f(\xi, \cdot)\|_{L_\tau^2} \left(\int < \tau - |\xi|^2 >^{-1-\varepsilon} d\tau \right)^{1/2} \right\|_{L_\xi^2} \lesssim \|f\|_{L_{\xi,\tau}^2}. \end{aligned}$$

□

We now turn to estimating the quintic nonlinearity $F_{quintic}$, i.e. estimate (14).

Proof. By symmetry, it suffices to obtain estimates only for $\| |\nabla \beta|^2 u_1 \|_{X_{100\varepsilon, -1/2+2\varepsilon}}$ in (12). We first perform a dyadic decomposition on $\nabla \beta$ and $\nabla \bar{\beta}$ to get

$$|\nabla \beta|^2 u_1 = \sum_{r_1, r_2} S_{r_1}(\nabla \beta) S_{r_2}(\nabla \bar{\beta}) u_1.$$

We split into two pieces, $\max(r_1, r_2) \leq 1$ and $\max(r_1, r_2) \geq 1$.

For the small frequency case, the argument goes along the lines of the estimate for $S(h, w, v)$ in (35). Observe that

$$S_{r_1}(\nabla \beta) S_{r_2}(\nabla \bar{\beta}) = S_{\max(r_1, r_2)}(S_{r_1}(\nabla \beta) S_{r_2}(\nabla \bar{\beta})).$$

Therefore, just as in the estimate for $S(h, w, v)$

$$\begin{aligned}
& \sum_{r_1, r_2 \leq 1} \|S_{r_1}(\nabla\beta)S_{r_2}(\nabla\bar{\beta})u_1\|_{X_{100\varepsilon, -1/2+2\varepsilon}} \lesssim \\
& \lesssim \sum_{r_1, r_2 \leq 1, R} \|S_{r_1}(\nabla\beta)S_{r_2}(\nabla\bar{\beta})S_R(u_1)\|_{X_{100\varepsilon, -1/2+2\varepsilon}} \lesssim \sum_{r_1, r_2 \leq 1, R} (R \max(r_1, r_2))^{2\varepsilon} \times \\
& \times \min(1, \left(\frac{\max(r_1, r_2)}{R}\right)^{1/2}) \|S_{r_1}(\nabla\beta)S_{r_2}(\nabla\bar{\beta})\|_{L^2} \|S_R(u_1)\|_{X_{100\varepsilon, 1/2+\varepsilon}} \lesssim \\
& \lesssim \|\nabla\beta\|_{L^4(\mathbb{R}^2 \times \mathbb{R}^1)}^2 \|u_1\|_{X_{100\varepsilon, 1/2+\varepsilon}}.
\end{aligned}$$

To estimate $\|\nabla\beta\|_{L^4(\mathbb{R}^2 \times \mathbb{R}^1)}$, one uses Sobolev embedding with one derivative in the space variable. We get

$$\|\nabla\beta\|_{L_x^4(\mathbb{R}^2)} \lesssim \|\nabla^2\beta\|_{L_x^{4/3}(\mathbb{R}^2)}.$$

From the definition of β , the boundedness of the Riesz transforms and the bilinear estimate (20), we have

$$\|\nabla^2\beta\|_{L_t^4 L_x^{4/3}} \lesssim \|\nabla^2 \Delta^{-1}(u_1 \bar{u}_2)\|_{L_t^4 L_x^{4/3}} \lesssim \|u_1 \bar{u}_2\|_{L_t^4 L_x^{4/3}} \lesssim \|u_1\|_{X_{\varepsilon, 1/2+\varepsilon}} \|u_2\|_{X_{\varepsilon, 1/2+\varepsilon}}.$$

Thus

$$\|\nabla\beta\|_{L^4(\mathbb{R}^2 \times \mathbb{R}^1)} \lesssim \|u_1\|_{X_{\varepsilon, 1/2+\varepsilon}} \|u_2\|_{X_{\varepsilon, 1/2+\varepsilon}}$$

and we have shown (12) for the quintic nonlinearity in the small frequency case.

For the large frequency case, we will have to show just as in the small frequency case

$$\sum_{r_1, r_2: \max(r_1, r_2) \geq 1, R} \|S_{r_1}(\nabla\beta)S_{r_2}(\nabla\bar{\beta})S_R(u_1)\|_{X_{100\varepsilon, -1/2+2\varepsilon}} \lesssim \|u_1\|_{X_{100\varepsilon, 1/2+\varepsilon}}^3 \|u_2\|_{X_{100\varepsilon, 1/2+\varepsilon}}^2$$

To verify that, following the argument in Proposition 1 and (35) with $v = S_{r_1}(\nabla\beta)S_{r_2}(\nabla\bar{\beta})$, we will have to demonstrate some decay in $\max(r_1, r_2)$ for $\|S_{r_1}(\nabla\beta)S_{r_2}(\nabla\bar{\beta})\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^1)}$. More precisely, we need to show

$$(36) \quad \|S_{r_1}(\nabla\beta)S_{r_2}(\nabla\bar{\beta})\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^1)} \lesssim \max(r_1, r_2)^{-\sigma} \|u_1\|_{X_{100\varepsilon, 1/2+\varepsilon}}^2 \|u_2\|_{X_{100\varepsilon, 1/2+\varepsilon}}^2,$$

for some $\sigma > 5\varepsilon$. By Cauchy-Schwartz (36) reduces to proving

$$(37) \quad \|S_M(\nabla\beta)\|_{L^4(\mathbb{R}^2 \times \mathbb{R}^1)} \lesssim M^{-\sigma} \|u_1\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|u_2\|_{X_{100\varepsilon, 1/2+\varepsilon}}.$$

By Sobolev embedding performed in the spatial variable only, the boundedness of the Riesz transforms and the definition of β , we get

$$\|S_M(\nabla\beta)\|_{L_x^4(\mathbb{R}^2)} \lesssim \|\nabla^2 S_M(\beta)\|_{L_x^{4/3}(\mathbb{R}^2)} \lesssim \|S_M(u_1 \bar{u}_2)\|_{L_x^{4/3}}.$$

Thus,

$$\|S_M(\nabla\beta)\|_{L_{x,t}^4} \lesssim \|\nabla^2 S_M(\beta)\|_{L_t^4 L_x^{4/3}(\mathbb{R}^2)} \lesssim \|S_M(u_1 \bar{u}_2)\|_{L_t^4 L_x^{4/3}},$$

and by the bilinear estimate (20), we get

$$\begin{aligned}
& \|S_M(\nabla\beta)\|_{L_{x,t}^4} \lesssim \max(\|S_M(u_1)\|_{X_{\varepsilon, 1/2+\varepsilon}} \|u_2\|_{X_{\varepsilon, 1/2+\varepsilon}}, \|S_M(u_2)\|_{X_{\varepsilon, 1/2+\varepsilon}} \|u_1\|_{X_{\varepsilon, 1/2+\varepsilon}}) \lesssim \\
& \lesssim M^{-99\varepsilon} \|u_1\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|u_2\|_{X_{100\varepsilon, 1/2+\varepsilon}}.
\end{aligned}$$

□

7. NULL FORM: ESTIMATES AWAY FROM THE DIAGONAL

This is the case when the “null” form is under control in the L^∞ norm. By symmetry, it suffices to consider the case $|z| \leq |\eta|/2$. Thus, we have the estimate

$$\frac{\langle \eta, z^\perp \rangle}{|\eta|^2} \lesssim \frac{|z|}{|\eta|}.$$

Also,

$$(38) \quad \Lambda_{offdiag} = \sum_R \int S_R(u_1 \overline{u_2}) S_R(\tilde{G}) = \sum_{M,N,R} \int S_R((u_1)_M (\overline{u_2})_N) \tilde{G}_R d\eta d\mu,$$

where

$$\hat{\tilde{G}}(\eta, \mu) = \int_{|z| \leq |\eta|/2} \frac{\langle \eta, z^\perp \rangle}{|\eta|^2} \frac{f(\eta - z, \mu - s) \chi_{\langle \eta - z \rangle^{100\varepsilon}}}{\chi_{\langle \mu - s - |\eta - z|^2 \rangle^{1/2-2\varepsilon}}} \frac{g(z, s)}{\chi_{\langle z \rangle^{100\varepsilon} \langle s - |z|^2 \rangle^{1/2+\varepsilon}}} dz ds.$$

It is clear now that every term in the dyadic formula (38) can be estimated by corresponding term for the form H from Proposition 1 times r/R and with $\kappa = 100\varepsilon$, which makes the double summation in r, R even easier. We get

$$|\Lambda_{offdiag}| \lesssim \|h_1\|_2 \|h_2\|_2 \|f\|_2 \|g\|_2.$$

8. NULL FORM: DIAGONAL ESTIMATES

In this section, we decompose the regions of the integration in (26) in such a way as to accomodate the behavior of the “null” form. Let us first represent the integration region over η and z as a union of dyadic annuli of the form $|\eta| \sim |z| \sim R$. Denote $\theta = \angle(\eta, z)$. Observe that if $|\theta| > 1/100$, we can control the L^∞ norm of the “null” form in a similar manner as in Section 2, and that will do in that case. By symmetry, we further assume that $0 < \theta < 1/100$. We decompose in the angular variable θ in a dyadic manner as $\theta \rightarrow 0$. Observe that

$$\begin{aligned} |\eta - z|^2 &= (|\eta| - |z|)^2 + 2|\eta||z|(1 - \cos(\theta)) \sim (|\eta| - |z|)^2 + R^2\theta^2, \\ \langle \eta, z^\perp \rangle &= |\eta||z|\sin(\theta) \sim R^2\theta. \end{aligned}$$

Obviously the size of $|\eta| - |z|$ is important at this stage, so we make the following partition of the area of integration

$$\begin{aligned} \mathcal{A}_l &= \{(\eta, z) : ||\eta| - |z|| \sim 2^l R\theta\}, \quad l \geq 1, \\ \mathcal{A}_0 &= \{(\eta, z) : ||\eta| - |z|| \leq R\theta\}. \end{aligned}$$

We will concentrate on the set \mathcal{A}_0 and in the end we will explain how to obtain similar estimates when integrating on \mathcal{A}_l with the corresponding exponential decay in l .

8.1. **The “really diagonal” case.** To summarize, we aim at controlling the expression

$$(39) \quad \sum_{R, \theta} \int_{|\eta|, |z| \sim R, ||\eta| - |z|| \leq R\theta} \frac{h_1(\xi - \eta + z, \tau - \mu) \langle \xi - \eta + z \rangle^{-100\varepsilon}}{\langle \tau - \mu - |\xi - \eta + z|^2 \rangle^{1/2+\varepsilon}} \frac{h_2(-\xi, -\tau) \langle \xi \rangle^{-100\varepsilon}}{\langle \tau - |\xi|^2 \rangle^{1/2+\varepsilon}} \\ \frac{\langle \eta, z^\perp \rangle}{|\eta - z|^2 \langle \mu - s - |\eta|^2 \rangle^{1/2-2\varepsilon}} \frac{f(\eta, \mu - s) \langle \eta \rangle^{100\varepsilon}}{\langle s - |z|^2 \rangle^{1/2+\varepsilon}} \frac{g(-z, s) \langle z \rangle^{-100\varepsilon}}{\langle s - |z|^2 \rangle^{1/2+\varepsilon}} d\xi d\eta dz d\tau d\mu ds.$$

Note that

$$\left| \frac{\langle \eta, z^\perp \rangle}{|\eta - z|^2} \right| \sim \theta^{-1}, \\ \langle \eta \rangle^{100\varepsilon} \langle z \rangle^{-100\varepsilon} \lesssim 1,$$

whenever $(\eta, z) \in \mathcal{A}_0$.

Take a partition the annulus $|\eta| \sim R$ into a finite intersection family of cubes with sides $R\theta$

$$\{|\eta| \sim R\} = \bigcup_{\nu} Q_{\nu}(\eta_{\nu}, R\theta).$$

By our assumptions on η and z , it is clear that whenever $\eta \in Q_{\nu_0}$, then $z \in Q_{\nu_0}^*$, where for a cube Q we denote by Q^* the cube with the same center and four times longer sides.

The Schur's test (cf. Lemma 3.11, [8]) and the finite intersection property of the family $\{Q_{\nu}\}_{\nu}$ imply

$$\left\| \chi_{\{(\eta, z) \in \mathcal{A}_0\}} \right\|_{[4, R^{2+1}]} \lesssim \left\| \sum_{\nu} \chi_{\{\eta \in Q_{\nu}, z \in Q_{\nu}^*\}} \right\|_{[4, R^{2+1}]} \lesssim \sup_{\nu} \left\| \chi_{\{\eta, z \in Q_{\nu}^*\}} \right\|_{[4, R^{2+1}]}.$$

Thus, we may further restrict the region of integration in (39) to a given cube $Q_0(\eta_0, R\theta)$. By the box localization principle (Lemma 5), since we have managed to restrict two of the variables to a box of size $R\theta$, we may do so for the other two variables as well. Thus, we are lead to consider multipliers of the form $\chi_{\{(\eta, z) \in Q_0(\eta_0, R\theta), \xi \in B(\xi_0, R\theta)\}}$, where $|\eta_0| \sim R$ and $\xi_0 \in \mathbb{R}^2$. It is worth mentioning that η_0 and ξ_0 may depend on R, θ . Denote

$$L_1 = \tau - \mu - |\xi - \eta + z|^2, \quad L_2 = \tau - |\xi|^2; \\ L_3 = \mu - s - |\eta|^2, \quad L_4 = s - |z|^2.$$

To control (39), we need to show that

$$(40) \quad \sum_{R, \theta} \sup_{\substack{\eta_0, \xi_0 \in \mathbb{R}^2 \\ |\eta_0| \sim R}} \frac{\max(\langle |\xi_0| \rangle, \langle R\theta \rangle)^{-100\varepsilon}}{\theta} \times \\ \times \left\| \frac{\chi_{\{\eta, z \in Q_0(\eta_0, R\theta), |\xi - \xi_0| \leq R\theta\}}}{(\langle L_1 \rangle \langle L_2 \rangle \langle L_4 \rangle)^{1/2+\varepsilon} \langle L_3 \rangle^{1/2-2\varepsilon}} \right\| \lesssim 1,$$

where we have used the fact that $\langle \xi - \eta + z \rangle \langle \xi \rangle \geq \max(\langle |\xi_0| \rangle, \langle R\theta \rangle)$.

Next, we write the equivalent quadrilinear form Λ_0 representing the multiplier in (40).

$$(41) \quad \Lambda_0(h_1, h_2, f, g) = \int \frac{h_1(\xi - \eta + z, \tau - \mu)}{\langle L_1 \rangle^{1/2+\varepsilon}} \frac{h_2(-\xi, -\tau)}{\langle L_2 \rangle^{1/2+\varepsilon}} \\ \times \frac{f(\eta, \mu - s)}{\langle L_3 \rangle^{1/2-2\varepsilon}} \frac{g(-z, s)}{\langle L_4 \rangle^{1/2+\varepsilon}} d\xi d\eta dz d\tau d\mu ds, \\ \begin{aligned} & z, \eta \in Q_0(\eta_0, R\theta), \\ & |\xi - \xi_0| < R\theta \end{aligned}$$

The following lemma allows us to dramatically reduce the number of cases.

Lemma 9. *With the restrictions in the integration in (41), either $L_4^* \gtrsim R^2$ or $|\xi_0| \gtrsim R/\theta$.*

Proof. Assume otherwise. Then $R^2 \gg |L_3 + L_4| = |\mu - |\eta|^2 - |z|^2|$. Since $|\eta|^2 + |z|^2 \sim R^2$, it follows that $\mu \sim R^2$.

On the other hand $|\mu - |\xi|^2 + |\xi - \eta + z|^2| = |L_2 - L_1| \ll R^2$. But $||\xi - \eta + z|^2 - |\xi|^2| \leq |\eta - z|(|\xi| + |\xi - \eta + z|) \lesssim R\theta \max(R\theta, |\xi_0|) \ll R^2$. Thus $\mu \ll R^2$, a contradiction. \square

We will actually show that

$$\sup_{\eta_0, \xi_0: |\eta_0| \sim R} |\Lambda_0(h_1, h_2, f, g)| \leq C_{R, \theta, \xi_0},$$

for suitable C_{R, θ, ξ_0} , such that

$$(42) \quad \sum_{R, \theta} \sup_{\xi_0} \frac{\max(\langle |\xi_0| \rangle, \langle R\theta \rangle)^{-100\varepsilon}}{\theta} C_{R, \theta, \xi_0} \lesssim 1.$$

Case 1. $L_4^* \gtrsim R^2$.

A subcase that can be easily handled is when $L_3^* \gtrsim R^2\theta$.

• Case 1.1. $L_3^* \gtrsim R^2\theta$

Proof. Observe that the multiplier from (41) has the form

$$(43) \quad \chi \left\{ \begin{array}{l} (\tau_i, \xi_i), \xi_i \in Q_i(\xi_i^0, R\theta), \\ \tau_i \pm |\xi_i|^2 \sim L_i \end{array} \right\}$$

where $Q_i(\xi_i^0, R\theta)$ are cubes with sidelength $R\theta$. Since all the variables are well localized, we can use (28) to estimate the $[4, R^{2+1}]$ norm of the multiplier in (43). Indeed, let us assume for simplicity that $L_1 = L_1^*, L_2 = L_2^*$. Then choose A, B in (28) so that

$$\begin{aligned} A &= \{(\tau_1, \xi_1) : |\xi_1 - \xi_1^0| \leq R\theta, \tau_1 - |\xi_1|^2 \sim L_1\}, \\ B &= \{(\tau_2, \xi_2) : |\xi_2 - \xi_2^0| \leq R\theta, \tau_2 + |\xi_2|^2 \sim L_2\}. \end{aligned}$$

For fixed ξ_1, ξ_2, τ_1 and τ_2 span intervals of length L_1 and L_2 respectively. Therefore, since ξ_1, ξ_2 are both within a ball with radius $R\theta$, we obtain from (28)

$$(44) \quad \left\| \chi \left\{ \begin{array}{l} (\tau_i, \xi_i), \xi_i \in Q_i(\xi_i^0, R\theta), \\ \tau_i \pm |\xi_i|^2 \sim L_i \end{array} \right\} \right\|_{[4, R^{2+1}]} \lesssim (L_1^* L_2^*)^{1/2} R^2 \theta^2.$$

Based on (44), we have

$$|\Lambda_0(h_1, h_2, f, g)\chi_{L_3^* \gtrsim R^2\theta}| \lesssim \sum_{L_4^* \gtrsim R^2, L_3^* \gtrsim R^2\theta, L_1^*, L_2^*} \frac{R^2\theta^2}{(< L_1^* > < L_2^* >)^\varepsilon L_4^{*1/2-2\varepsilon} L_3^{*1/2+\varepsilon}} \lesssim R^{2\varepsilon}\theta^{3/2-\varepsilon},$$

which implies (42). \square

• **Case 1.2.** $L_3^* \lesssim R^2\theta$.

To avoid the enormous amount of cases to consider, we make the following reduction. Observe that L_3 and L_4 appear symmetrically (they are both of the type $\tau_i - |\xi_i|^2$), except in the power that they have in the denominator. Thus, since L_3 appear with a lesser power, it will be enough to consider the case $L_3 \geq L_4$, that is L_4^* does not fall into L_4 itself. With this reduction Case 1.2 breaks into five different subcases. More precisely, we subdivide Case 1.2 into

- Case 1.2.1 $L_4^* = L_4$ or L_2 .
- Case 1.2.2 $(L_1, L_2) = (L_1^*, L_2^*)$ or (L_2^*, L_1^*) ; $|\xi_0| \approx R$ and $L_3 = L_4^*$.
- Case 1.2.3 $(L_2, L_4) = (L_1^*, L_2^*)$ or (L_2^*, L_1^*) ; $|\xi_0| \approx R$ and $L_3 = L_4^*$.
- Case 1.2.4 $(L_1, L_4) = (L_1^*, L_2^*)$ or (L_2^*, L_1^*) ; $|\xi_0| \approx R$ and $L_3 = L_4^*$.
- Case 1.2.5 $|\xi_0| \sim R$ and then we are considering
 - Case 1.2.5a) $\theta > 1/R$
 - Case 1.2.5.b) $\theta < 1/R$ since the relative sizes of $R\theta$ and ξ will matter.

We first dispose of the case, when $L_4^* = L_1$ or $L_4^* = L_2$.

Case 1.2.1. $L_4^* = L_1$ or $L_4^* = L_2$, $L_3^* \lesssim R^2\theta$, $L_4^* \gtrsim R^2$.

Proof. An application of the Cauchy-Schwartz yields

$$\Lambda_0(h_1, h_2, f, g) \lesssim \sup_{\|h\|_2=1} |\Lambda_1(h_1, h_2, h)| \sup_{\|h\|_2=1} |\Lambda_2(f, g, h)|,$$

where

$$\Lambda_1(h_1, h_2, h) = \int_{\substack{|\xi - \xi_0| \lesssim R\theta, \\ |\tilde{\eta}| \lesssim R\theta}} \frac{h_1(\xi - \tilde{\eta}, \tau - \mu)}{< \tau - \mu - |\xi - \tilde{\eta}|^2 >^{1/2+\varepsilon}} \frac{h_2(-\xi, -\tau)}{< \tau - |\xi|^2 >^{1/2+\varepsilon}} h(\tilde{\eta}, \mu) d\xi d\tau d\tilde{\eta} d\mu$$

and

$$\Lambda_2(f, g, h) = \int_{\substack{|\eta - \eta_0| \lesssim R\theta, \\ |z - \eta_0| \lesssim R\theta}} \frac{f(\eta, \mu - s)}{< \mu - s - |\eta|^2 >^{1/2-2\varepsilon}} \frac{g(-z, s)}{< s - |z|^2 >^{1/2+\varepsilon}} h(z - \eta, -\mu) dz d\eta d\mu ds.$$

Observe that by Lemma 6, we can estimate

$$(45) \quad |\Lambda_2| \lesssim \sum_{L_3, L_4 \ll R^2\theta} \frac{(L_4)^{1/2} (R\theta/R)^{1/2} L_3^{1/2}}{< L_3 >^{1/2-2\varepsilon} < L_4 >^{1/2+\varepsilon}} \lesssim R^{4\varepsilon} \theta^{1/2}.$$

By Lemma 7 we need to compute

$$|H| = ||\xi - \tilde{\eta}|^2 - |\xi|^2 + |\tilde{\eta}|^2| \lesssim |\tilde{\eta}|(|\xi| + |\tilde{\eta}|) \lesssim R\theta \max(R\theta, |\xi_0|)$$

Then the estimates are

$$(46) \quad |\Lambda_1| \lesssim \sum_{L_1, L_2, \max(L_1, L_2) \gtrsim R^2} \frac{\min(L_1, L_2)^{1/2} (R\theta / \max(R\theta, |\xi_0|))^{1/2} (R\theta \max(R\theta, |\xi_0|))^{1/2}}{(< L_1 > < L_2 >)^{1/2+\varepsilon}} \lesssim \frac{\theta}{R^{2\varepsilon}}.$$

Combining (45) and (46) gives (42) in Case 1.2.1. \square

We will postpone the somewhat peculiar case $|\xi_0| \sim R$ for later on.

We consider the case, where L_1 and L_2 are the two smallest numbers in the sequence L_1, L_2, L_3, L_4 .

Case 1.2.2 $(L_1, L_2) = (L_1^*, L_2^*)$ or (L_2^*, L_1^*) , $|\xi_0| \asymp R$, $L_3 = L_4^* \gtrsim R^2$.

Proof. In that case, we will fully use the quadrilinear form, instead of relying on Cauchy-Schwartz and then deal with the resulting trilinear forms. We estimate the multiplier in (41) by (28). We have an upper bound of

$$(47) \quad \sum_{L_1, L_2, L_3, L_4} \frac{\left| \left\{ (\tau_1, \xi_1), (\tau_2, \xi_2) \in (\Omega_1 \times \Omega_2) : (\tilde{\tau} - \tau_1 - \tau_2, \tilde{\xi} - \xi_1 - \xi_2) \in \Omega_4 \right\} \right|^{1/2}}{(< L_1 > < L_2 > < L_4 >)^{1/2+\varepsilon} < L_3 >^{1/2-2\varepsilon}},$$

where $\tilde{\tau}, \tilde{\xi}$ are fixed and

$$\begin{aligned} \Omega_1 &= \{(\tau_1, \xi_1) : \tau_1 - |\xi_1|^2 \sim L_1, \quad \xi_1 = \xi - \eta + z, |\xi - \xi_0| \leq R\theta, \quad |\eta - z| \leq R\theta\} \\ \Omega_2 &= \{(\tau_2, \xi_2) : \tau_2 + |\xi_2|^2 \sim L_2, \quad \xi_2 = -\xi, \quad |\xi - \xi_0| \leq R\theta\} \\ \Omega_4 &= \{(\tau_4, \xi_4) : \tau_4 - |\xi_4|^2 \sim L_4, \quad \xi_4 = -z, \quad |z - z_0| \leq R\theta\} \end{aligned}$$

Note that for a fixed spatial variables the time variables span intervals of length L_1 and L_2 respectively. Also, we have

$$(48) \quad \tilde{\tau} - |\tilde{\xi} - \xi_1 - \xi_2|^2 - |\xi_1|^2 + |\xi_2|^2 = L_1 + L_2 + L_4.$$

For a **fixed** ξ_2 , we have (based on (48))

$$|\tilde{\xi} - \xi_1 - \xi_2|^2 + |\xi_1|^2 = \text{const} + O(L_3^*).$$

By the parallelogram law,

$$(49) \quad |(\tilde{\xi} - \xi_2)/2 - \xi_1|^2 = \text{const} + O(L_3^*)$$

Furthermore, since $\tilde{\xi} - (\xi_1 + \xi_2) \in B(z_0, R\theta)$ and $|\xi_1 + \xi_2| \lesssim R\theta$, we infer $\tilde{\xi} \sim R$. Thus taking into account that $|\xi_0| \asymp R$, we conclude that $(\tilde{\xi} - \xi_2)/2 - \xi_1 \sim \max(R, |\xi_0|)$ and therefore by (49), ξ_1 is contained in an annulus with radius $\max(R, |\xi_0|)$ and thickness $L_3^*/\max(R, |\xi_0|)$. Observe that ξ_1 is also in a ball with radius $R\theta$, therefore it belongs to a rectangle with sides $R\theta$ and $L_3^*/\max(R, |\xi_0|)$. Finally, since ξ_2 belongs to a ball with radius $R\theta$, one estimates (47) by

$$\sum_{L_1, L_2, L_3, L_4} \frac{(L_1 L_2)^{1/2} R\theta \left(\frac{L_3^* R\theta}{\max(R, |\xi_0|)} \right)^{1/2}}{(< L_1 > < L_2 > < L_4 >)^{1/2+\varepsilon} L_3^{1/2-2\varepsilon}} \lesssim R^{4\varepsilon} \theta^{3/2},$$

thus implying (42). \square

Case 1.2.3 $(L_2, L_4) = (L_1^*, L_2^*)$ or (L_2^*, L_1^*) , $|\xi_0| \asymp R$, $L_3 = L_4^*$.

Proof. This case is very similar to Case 1.2.2. We estimate the multiplier by

$$(50) \quad \sum_{L_1, L_2, L_3, L_4} \frac{\left| \left\{ \{(\tau_1, \xi_1), (\tau_2, \xi_2)\} \in (\Omega_2 \times \Omega_4) : (\tilde{\tau} - \tau_1 - \tau_2, \tilde{\xi} - \xi_1 - \xi_2) \in \Omega_1 \right\} \right|^{1/2}}{(< L_1 > < L_2 > < L_4 >)^{1/2+\varepsilon} < L_3 >^{1/2-2\varepsilon}},$$

where $\Omega_1, \Omega_2, \Omega_4$ are the sets defined before. We have

$$\tilde{\tau} - |\tilde{\xi} - \xi_1 - \xi_2|^2 - |\xi_2|^2 + |\xi_1|^2 = L_1 + L_2 + L_4 = O(L_3^*)$$

For **fixed** ξ_1 , we have by the parallelogram law

$$(51) \quad |(\tilde{\xi} - \xi_1)/2 - \xi_2| = \text{const} + O(L_3^*).$$

Since $\tilde{\xi} - \xi_1 - \xi_2 \in B(\xi_0, R\theta)$ and $|\xi_1 + \xi_0| \lesssim R\theta$, it follows that $|\tilde{\xi} - \xi_2| \lesssim R\theta$. In particular, after taking into account that $\xi_0 \approx R$, we obtain $|(\tilde{\xi} - \xi_1)/2 - \xi_2| \sim \max(R, |\xi_0|)$. By (51), one has that ξ_2 is contained in an annulus with radius $\max(R, |\xi_0|)$ and thickness $L_3^*/\max(R, |\xi_0|)$. Since ξ_2 is also contained in a ball with radius $R\theta$, we have that ξ_2 is contained in a rectangle with sidelengths $R\theta$ and $L_3^*/\max(R, |\xi_0|)$ for every fixed ξ_1 . The usual observation that τ_1 and τ_2 sweep intervals of length L_2 and L_4 respectively, leads us to estimate (50) by

$$\sum_{L_1, L_2, L_3, L_4} \frac{(< L_2 > < L_4 >)^{1/2} R\theta \left(\frac{L_3^* R\theta}{\max(R, |\xi_0|)} \right)^{1/2}}{(< L_1^* > < L_2^* > < L_3^* >)^{1/2+\varepsilon} L_4^{1/2-2\varepsilon}} \lesssim R^{4\varepsilon} \theta^{3/2},$$

which again implies (42). \square

Case 1.2.4 $(L_1, L_4) = (L_1^*, L_2^*)$ or (L_2^*, L_1^*) , $|\xi_0| \approx R$, $L_3 = L_4^* \gtrsim R^2$.

Proof. We estimate the norm of the multiplier by

$$(52) \quad \sum_{L_1, L_2, L_3, L_4} \frac{\left| \left\{ \{(\tau_1, \xi_1), (\tau_2, \xi_2)\} \in (\Omega_1 \times \Omega_4) : (\tilde{\tau} - \tau_1 - \tau_2, \tilde{\xi} - \xi_1 - \xi_2) \in \Omega_2 \right\} \right|^{1/2}}{(< L_1 > < L_2 > < L_4 >)^{1/2+\varepsilon} < L_3 >^{1/2-2\varepsilon}},$$

where $\Omega_1, \Omega_2, \Omega_4$ are the sets defined in Case 1.2.2. Like in the previous cases, we have a relation involving some of the variables. Here, we have

$$\tilde{\tau} - |\xi_1|^2 - |\xi_2|^2 + |\tilde{\xi} - \xi_1 - \xi_2|^2 = L_1 + L_2 + L_4 = O(L_3^*).$$

We change variables $\lambda_1 = \xi_1 + \xi_2$, $\lambda_2 = \xi_1 - \xi_2$ and we are interested in the measure of the corresponding set in (52). Since the Jacobian of the transformation is two, we pass to the new variables. Fix λ_1 . Observe also that since $|\xi_0| \approx R$, $|\lambda_2| \sim \max(R, |\xi_0|)$. We have then by the parallelogram law

$$\begin{aligned} |\xi_1|^2 + |\xi_2|^2 &= \text{const} + O(L_3^*), \\ |\xi_1 - \xi_2|^2 &= \text{const} + O(L_3^*). \end{aligned}$$

That implies that for fixed λ_1 , λ_2 is contained in an annulus with thickness $L_3^*/\max(R, |\xi_0|)$. On the other hand, λ_2 is contained in a ball with radius $R\theta$. These estimates, together with

the usual observations that τ_1, τ_2 are in intervals of length L_1 and L_4 respectively, and the fact that λ_1 sweeps a ball with radius $R\theta$ imply the following bound for (52)

$$\sum_{L_1, L_2, L_3, L_4} \frac{(L_1 L_4)^{1/2} R\theta \left(\frac{L_3^* R\theta}{\max(R, |\xi_0|)} \right)^{1/2}}{(< L_1^* > < L_2^* > < L_3^* >)^{1/2+\varepsilon} L_4^{*1/2-2\varepsilon}} \lesssim R^{4\varepsilon} \theta^{3/2},$$

which clearly implies (42). \square

Finally, we deal with the case $|\xi_0| \sim R$.

•**Case 1.2.5** $|\xi_0| \sim R, L_4^* \gtrsim R^2, L_3^* \lesssim R^2 \theta$

In that case, the relative size of $R\theta$ and ξ_0 will matter, so we will split into two subcases.

Case 1.2.5a) $\theta > 1/R$

Remark This case is vacuous if $R < 1$.

Proof. Apply Cauchy-Schwartz to Λ_0 obtain

$$|\Lambda_0(h_1, h_2, f, g)| \lesssim \sup_{\|h\|_2=1} |\Lambda_1(h_1, h_2, h)| \sup_{\|h\|_2=1} |\Lambda_2(f, g, h)|,$$

where Λ_1 and Λ_2 were defined in Case 1.2. Compute again $H = \|\tilde{\eta}\|^2 + |\xi - \tilde{\eta}|^2 - |\xi|^2 \lesssim R^2 \theta$. Thus, based on the estimates in Lemma 7, we conclude that

$$|\Lambda_1| \lesssim \sum_{L_1, L_2} \min(L_1, L_2)^{1/2} (R\theta/R)^{1/2} \frac{\min(R^2 \theta, \max(L_1, L_2)/\theta)^{1/2}}{(< L_1 > < L_2 >)^{1/2+\varepsilon}}$$

and by the estimate for m_1 in Lemma 6

$$|\Lambda_2| \lesssim \sum_{L_3, L_4} \frac{\min(L_3, L_4)^{1/2} (R\theta/R)^{1/2} \min(\max(L_3, L_4), R^2 \theta)^{1/2}}{< L_3 >^{1/2-2\varepsilon} < L_4 >^{1/2+\varepsilon}}.$$

We have the estimate (after quickly going through the appropriate cases - L_4^* is either L_1 or L_2 or L_4^* is either L_3 or L_4)

$$|\Lambda_1| |\Lambda_2| \lesssim R^{4\varepsilon} \theta.$$

Thus we can sum up the expression in (42) as follows

$$\sum_R \sum_{\theta > 1/R, \theta \text{ dyadic}} \frac{R^{-100\varepsilon}}{\theta} R^{4\varepsilon} \theta \lesssim \sum_R R^{-96\varepsilon} \ln R \lesssim 1$$

\square

Case 1.2.5b) $\theta < 1/R$.

Proof. We concentrate on the high frequency case $R > 1$. The case $R < 1$ is trivial, because then $< L_1^* > = < L_2^* > = < L_3^* > = 1$ and one easily estimates (see estimates below). For simplicity, we assume once again that $L_3 = L_4^* \gtrsim R^2$. Observe that L_3 appears with the smallest power in the denominator and that should be the worst case for the maximum to occur. Moreover later in the proof regarding that case, we will see that we could perform the

same argument with any other configuration of $L_1^*, L_2^*, L_3^*, L_4^*$. We use again the quadrilinear form Λ_0 . By (28), we get

$$(53) \quad \sum_{L_1, L_2, L_3, L_4} \frac{\left| \left\{ (\tau_1, \xi_1), (\tau_2, \xi_2) \in (\Omega_1 \times \Omega_4) : (\tilde{\tau} - \tau_1 - \tau_2, \tilde{\xi} - \xi_1 - \xi_2) \in \Omega_2 \right\} \right|^{1/2}}{(< L_1 > < L_2 > < L_4 >)^{1/2+\varepsilon} < L_3 >^{1/2-2\varepsilon}}.$$

We have the relation

$$(54) \quad \tilde{\tau} + |\tilde{\xi} - \xi_1 - \xi_2|^2 - |\xi_1|^2 - |\xi_2|^2 = L_1 + L_2 + L_4 = O(L_3^*).$$

There are two distinct possibilities now. Either $|\xi_0 + z_0| \geq R$ or $|\xi_0 - z_0| \geq R$ (or both). We show the desired estimate, for the case $|\xi_0 + z_0| \geq R$, the other case being similar. Observe that $|\xi_1 - \xi_2| = |\xi_0 + z_0| + O(R\theta) \gtrsim R$. We introduce again the new variables $\lambda_1 = \xi_1 + \xi_2, \lambda_2 = \xi_1 - \xi_2$ and we fix λ_1 . The parallelogram law and (54) imply

$$\begin{aligned} |\xi_1|^2 + |\xi_2|^2 &= \text{const} + O(L_3^*), \\ |\xi_1 - \xi_2|^2 &= \text{const} + O(L_3^*) \end{aligned}$$

and thus, we have that for fixed λ_1, λ_2 is contained in an annulus with thickness L_3^*/R . On the other hand it is contained in a ball with radius $R\theta$. The usual observation that τ_1, τ_2 span intervals of length L_1, L_4 , gives us the following estimate for (53)

$$\sum_{L_3 \gtrsim R^2, L_3^* \ll R^2\theta} \frac{(L_1 L_4)^{1/2} R \theta (R \theta L_3^*/R)^{1/2}}{(< L_1^* L_2^* L_3^* >)^{1/2+\varepsilon} < L_3 >^{1/2-2\varepsilon}} \lesssim \sum_{L_3^* \ll R^2\theta} \theta (L_3^* \theta)^{1/2} R^{4\varepsilon} \lesssim R \theta^2 R^{4\varepsilon},$$

which implies

$$\sum_R \sum_{\theta < 1/R} \frac{R^{-100\varepsilon} R \theta^2 R^{4\varepsilon}}{\theta} \lesssim 1,$$

which is the desired estimate (42). \square

Now, we pass to the other possibility alluded to in Lemma 9, namely that $|\xi_0| \gtrsim R/\theta$. Since we have exhausted the cases, where $L_4^* \gtrsim R^2$, we will consider only $L_4^* \ll R^2$.

Case 2. $|\xi_0| > R/\theta, L_4^* \ll R^2$.

Proof. We will have to apply Cauchy-Schwartz's inequality with a reorganized pairs of functions. We do that in order to take advantage of the disparity in the sizes of $|\xi_0|$ and $|\eta_0|$.

$$\Lambda_0(h_1, h_2, f, g) \lesssim \sup_{\|h\|_2=1} |\Lambda_1(h_1, h_2, h)| \sup_{\|h\|_2=1} |\Lambda_2(f, g, h)|,$$

where

$$\begin{aligned} \Lambda_1(h_1, h_2, h) &= \int \frac{h_1(\xi, \tau_1)}{< \tau_1 - |\xi|^2 >^{1/2+\varepsilon}} \frac{f(\eta, \tau_2)}{< \tau_2 - |\eta|^2 >^{1/2-2\varepsilon}} \times \\ &\quad \begin{aligned} |\xi - \xi_0| &\lesssim R\theta, \\ |\eta - \eta_0| &\lesssim R\theta \\ L_4^* &\ll R^2 \end{aligned} \\ &\times h(-\xi - \eta, -\tau_1 - \tau_2) d\xi d\eta d\tau_1 d\tau_2 \end{aligned}$$

and

$$\begin{aligned} \Lambda_2(f, g, h) &= \int_{\substack{|z - \eta_0| \lesssim R\theta, \\ |\xi - \xi_0| \lesssim R\theta}} \frac{h_2(-\xi, -\tau_1)}{< \tau_1 - |\xi|^2 >^{1/2+\varepsilon}} \frac{g(z, \tau_2)}{< \tau_2 - |z|^2 >^{1/2+\varepsilon}} \times \\ &\quad \times h(\xi - z, -\tau_1 - \tau_2) d\xi dz d\tau_1 d\tau_2. \end{aligned}$$

For Λ_1 , we are in a position to use Lemma 6. Since $L_4^* \ll R^2$, we have

$$|\Lambda_1| \lesssim \sum_{L_1, L_3 \ll R^2} \frac{\min(L_1, L_3)^{1/2}}{L_1^{1/2+\varepsilon} L_3^{1/2-2\varepsilon}} \left(\frac{R}{R/\theta} \right)^{1/2} \max(L_1, L_3)^{1/2} \lesssim R^{4\varepsilon} \theta^{1/2}.$$

For Λ_2 , we use Lemma 8 to infer

$$|\Lambda_2| \lesssim \theta^{1/2}.$$

For $R > 1$, we combine the estimates for Λ_1 and Λ_2 to show

$$\sum_{R, \theta} \left(\frac{\theta}{R} \right)^{100\varepsilon} \frac{R^{4\varepsilon} \theta^{1/2} \theta^{1/2}}{\theta} \lesssim 1,$$

which is the desired inequality (42). For $R < 1$, the estimates above can be improved greatly and thus one estimates in that case as well. \square

8.2. Null form: The not so diagonal case. This is the case where the integration in the definition of Λ_0 is over the set \mathcal{A}_l . Note first, that if $(\eta, z) \in \mathcal{A}_l$, then

$$(55) \quad \left| \frac{\langle \eta, z^\perp \rangle}{|\eta - z|^2} \right| \sim \frac{1}{2^{2l}\theta}.$$

Thus, if $2^l\theta \gtrsim 2^{-l/2}$, the L^∞ norm of the “null” form is under control (with exponential decay in l) and we can estimate as in the off-diagonal case. So assume from now on that $2^l\theta \ll 2^{-l/2}$. Denote

$$\begin{aligned} \Lambda_l(h_1, h_2, f, g) &= \int_{(\eta, z) \in \mathcal{A}_l, |\eta|, |z| \sim R} \frac{h_1(\xi - \eta + z, \tau - \mu)}{< L_1 >^{1/2+\varepsilon}} \frac{h_2(-\xi, -\tau)}{< L_2 >^{1/2+\varepsilon}} \\ &\quad \times \frac{f(\eta, \mu - s)}{< L_3 >^{1/2-2\varepsilon}} \frac{g(-z, s)}{< L_4 >^{1/2+\varepsilon}} d\xi d\eta dz d\tau d\mu ds. \end{aligned}$$

Taking into account (55), we need to show that

$$(56) \quad \sum_{R, \theta, l: 2^l\theta \ll 2^{-l/2}} \sup_{\xi_0} \frac{\max(|\xi_0|, < 2^l R \theta >)^{-100\varepsilon}}{2^{2l}\theta} |\Lambda_l| \lesssim 1.$$

Partition the annulus

$$\{|\eta| \sim R\} = \bigcup_{\nu \in \Theta} Q(\eta_\nu, R\theta),$$

into a finite intersection family of cubes of sidelength $R\theta$. Partition the set \mathcal{A}_l accordingly

$$\mathcal{A}_l = \bigcup_{j=-2^{l-1}}^{2^{l+1}} \{(\eta, z) \in \mathcal{A}_l : ||\eta| - |z| - (2^l + j)R\theta| \leq R\theta\} = \bigcup_j \mathcal{A}_l^j.$$

For every fixed j , there is a selector map $m_j : \Theta \rightarrow \Theta$, so that if $(\eta, z) \in \mathcal{A}_l^j$ and whenever $\eta \in Q(\eta_\nu, R\theta)$, then $z \in Q(\eta_{m_j(\nu)}, R\theta)$ and $\eta_\nu \| \eta_{m_j(\nu)}$. This is possible since $\theta = \angle(\eta, z) \ll 1$. Thus, by the Schur's test

$$(57) \quad \|\Lambda_l\|_{[4, R^{2+1}]} \lesssim \sum_{j=-2^{l-1}}^{2^{l+1}} \|\Lambda_l^j\|_{[4, R^{2+1}]} \lesssim 2^l \sup_j \|\Lambda_l^j\|_{[4, R^{2+1}]},$$

where Λ_l^j are in the form Λ_0 with the integration taken in the corresponding region \mathcal{A}_l^j . By the localization principle, applied to each one of the forms Λ_l^j , we can further restrict the ξ integration to a ball with center ξ_0 and radius $R\theta$. Thus, we are lead to estimate quadrilinear forms

$$(58) \quad \Lambda(h_1, h_2, f, g) = \int \frac{h_1(\xi - \eta + z, \tau - \mu)}{< L_1 >^{1/2+\varepsilon}} \frac{h_2(-\xi, -\tau)}{< L_2 >^{1/2+\varepsilon}} \\ \frac{f(\eta, \mu - s)}{< L_3 >^{1/2-2\varepsilon}} \frac{g(-z, s)}{< L_4 >^{1/2+\varepsilon}} d\xi d\eta dz d\tau d\mu ds. \\ \begin{aligned} &|\eta - \eta_0| \leq R\theta, |z - z_0| \leq R\theta, \\ &|\xi - \xi_0| \leq R\theta, |\eta_0 - z_0| \sim 2^l R\theta \\ &\eta_0 \| z_0; |\eta_0|, |z_0| \sim R \end{aligned}$$

Thus to show (56), it will suffice to obtain an estimate

$$|\Lambda(h_1, h_2, f, g)| \leq C_{l,R,\theta} \|h_1\|_2 \|h_2\|_2 \|f\|_2 \|g\|_2,$$

where $C_{l,R,\theta}$ satisfies

$$(59) \quad \sum_{R,\theta,l: 2^l\theta \ll 2^{-l/2}} \sup_{\xi_0} \frac{\max(|\xi_0|, < 2^l R\theta >)^{-100\varepsilon}}{2^l \theta} C_{l,R,\theta} \lesssim 1.$$

We start reviewing the proof that we gave for the boundedness of the similar quadrilinear form Λ_0 given in (41). First, observe that Lemma 9 has to read now

Lemma 10. $L_4^* \gtrsim R^2$ or $|\xi_0| \gtrsim R/(2^l\theta)$.

In the Case 1.1, we will have the exact same estimate regardless of the new restriction $|\eta_0 - \xi_0| \sim 2^l R\theta$, which will enable us to add up in (59) thanks to the exponential factor in the denominator.

In Case 1.2.1, we obtain the estimate $|\Lambda| \lesssim R^{2\varepsilon} (2^l\theta)^{3/2}$, rather than $|\Lambda| \lesssim R^{2\varepsilon} (\theta)^{3/2}$, but that still implies the validity of (59), since

$$\sum_{R,\theta,l: 2^l\theta \ll 2^{-l/2}} \frac{(2^l\theta)^{3/2} R^{2\varepsilon}}{2^l \theta (2^l R\theta)^{100\varepsilon}} \lesssim 1.$$

In Cases 1.2.2, 1.2.3, 1.2.4 we will have absolutely no change in the estimates, hence we can add up (in l) in (59), due to the exponential factor in the denominator.

The restrictions in Case 1.2.5a) should be changed now to $|\xi_0| \sim R, L_4^* \gtrsim R^2, L_3^* \lesssim R^2\theta, 2^l\theta > 1/R$. The case $R \leq 1$ is easy to estimate. Indeed, one can proceed as in Case 1.1 to estimate the multiplier norm in (58) by $R^2\theta^2$ and therefore (59) follows by

$$\sum_{l, R \leq 1, \theta \leq 1} \frac{R^2\theta^2}{2^l\theta} \lesssim 1.$$

For $R > 1$, a careful inspection of the argument, shows that one has an estimate $|\Lambda| \lesssim 2^l\theta R^{4\epsilon}$, but we can still add up in (59), since

$$\sum_{R, l, \theta: 2^l\theta \ll 2^{-1/2}} \frac{R^{-100\epsilon} R^{4\epsilon} 2^l\theta}{2^l\theta} \lesssim \sum_R R^{-96\epsilon} \sum_{l, \theta: 1 \leq l \lesssim \ln(R), \theta \gtrsim R^{-3}} \lesssim \sum_R \ln^2(R) R^{-96\epsilon} \lesssim 1.$$

In Case 1.2.5 b), we treat $|\xi_0| \sim R, L_4^* \gtrsim R^2, L_3^* \lesssim R^2\theta, 2^l\theta < 1/R$. The case $R < 1$ can be performed as in the case 1.2.5a) above. For $R > 1$, the argument in Case 1.2.5b) shows that the same estimate holds (regardless of the new restriction on η_0, ξ_0). Thus we add up with the help of the exponential factor in (59).

In Case 2 according to Lemma 10, we deal with $|\xi_0| \gtrsim R/(2^l\theta)$, instead of $|\xi_0| \gtrsim R/\theta$. For Case 2.1, the estimate is $|\Lambda| \lesssim R^{4\epsilon} 2^{l/2} \theta^{1+4\epsilon}$. Thus (59) is bounded by

$$\sum_{R, \theta, l: 2^l\theta \ll 1} \frac{R^{4\epsilon} \theta 2^{l/2}}{2^l\theta < R/(2^l\theta) >^{100\epsilon}} \lesssim \sum_{\theta, l: 2^l\theta \ll 1} \frac{(2^l\theta)^{100\epsilon}}{2^{l/2}} \lesssim 1.$$

Finally, in Case 2, the restrictions are $|\xi_0| \gtrsim R/(2^l\theta), L_4^* \ll R^2\theta^2$. We obtain an estimate $|\Lambda| \lesssim (2^l\theta)^{3/2} R^{6\epsilon}$, which is handled as in the Case 1.2.1.

9. REGULARITY RESULTS

In this section, we will show that once we have a local existence result for data $u_0 \in H^{100\epsilon}$ with lifespan for the solution $T = T(\|u_0\|_{H^{100\epsilon}})$, then we have local existence for data $u_0 \in H^k$ with a lifespan for the solution $T_k = T(\|u_0\|_{H^{100\epsilon}}, k)$. More precisely, we have

Theorem 5. (*Regularity estimates*)

For a given data $u_0 \in H^k$, the system (8) has an unique solution u defined at least for time $T = T(\|u_0\|_{H^{100\epsilon}}, k)$ and there exists a constant $C_{\epsilon, k}$, so that

$$\|u\|_{X_{k, 1/2+\epsilon}} \leq C_{\epsilon, k} \|u_0\|_{H^k}.$$

Proof. For our purposes it will suffice to check the statement of the theorem for some specific sequence (k_n) , so that $k_n \rightarrow \infty$, since for every $u_0 \in H^k$, we will find n so that $k_{n+1} \geq k_n$ and the solution has a lifespan at least T_{k_n} . One could obtain estimates for the indices in between by the Leibnitz rule for fractional differentiation in the $X_{s,b}$ spaces. We do not pursue these however since they are not necessary for our purposes.

Due to the nature of the estimates, it will be convenient to take $k = n + 100\epsilon$. We will show the theorem for a cubic nonlinearity, since the others are treated in the same way. The common between them is the (anti) linearity structure that all of them exhibit. Take

$k = 1 + 100\varepsilon$. Since $u_0 \in H^1 \subset H^{100\varepsilon}$, it is clear that a solution exists and it satisfies the integral equation (10). Differentiating (10) yields

$$\nabla_x u(x, t) = e^{it\Delta} u_0 + \int_0^t e^{i(t-\tau)\Delta} \nabla_x F_u(\tau, \cdot) d\tau.$$

Assume now that $F_u = F_{cubic}$ for simplicity. Note that by the linearity of F_{cubic} and the product rule

$$\nabla_x F_{cubic}(u_1, u_2, u_3) = F_{cubic}(\nabla u_1, u_2, u_3) + F_{cubic}(u_1, \nabla u_2, u_3) + F_{cubic}(u_1, u_2, \nabla u_3),$$

where $u_3 = u_1$ or u_2 as usual. Thus, by combining estimates (11) and (12), we obtain

$$\begin{aligned} \|u\|_{X_{1+100\varepsilon, 1/2+\varepsilon}} &\lesssim \|u_0\|_{H^{1+100\varepsilon}} + T^\varepsilon \|u_1\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|u_2\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|\nabla u_3\|_{X_{100\varepsilon, 1/2+\varepsilon}} + \dots \\ &\lesssim \|u_0\|_{H^{1+100\varepsilon}} + T^\varepsilon \|u\|_{X_{1+100\varepsilon, 1/2+\varepsilon}} \|u\|_{X_{100\varepsilon, 1/2+\varepsilon}}^2. \end{aligned}$$

Thus, taking

$$T^\varepsilon \|u\|_{X_{100\varepsilon, 1/2+\varepsilon}}^2 \lesssim T^\varepsilon \|u_0\|_{H^{100\varepsilon}}^2 \ll 1$$

will allow us to hide $T^\varepsilon \|u\|_{X_{1+100\varepsilon, 1/2+\varepsilon}} \|u\|_{X_{100\varepsilon, 1/2+\varepsilon}}^2$ and we get (for a time $T = T(\|u\|_{H^{100\varepsilon}})$)

$$\|u\|_{X_{1+100\varepsilon, 1/2+\varepsilon}} \lesssim \|u_0\|_{H^{1+100\varepsilon}}.$$

We can basically iterate this result in the following manner. For $k = 2 + 100\varepsilon$, we proceed as follows. Differentiate (10) once more. Then, we have two groups of terms. In the first, the derivatives fall on different u , for example $F_{cubic}(\nabla u_1, \nabla u_2, u_3)$, for the second we will have terms like $F_{cubic}(\nabla^2 u_1, u_2, u_3)$. All the terms are estimated by (12). We get

$$\begin{aligned} (60) \quad \|u\|_{X_{2+100\varepsilon, 1/2+\varepsilon}} &\lesssim \|u_0\|_{H^{2+100\varepsilon}} + T^\varepsilon \|u_1\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|u_2\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|\nabla^2 u_3\|_{X_{100\varepsilon, 1/2+\varepsilon}} + \dots \\ &+ T^\varepsilon \|\nabla u_1\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|\nabla u_2\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|u_3\|_{X_{100\varepsilon, 1/2+\varepsilon}} + \dots \\ &\lesssim \|u_0\|_{H^{2+100\varepsilon}} + T^\varepsilon \|u\|_{X_{2+100\varepsilon, 1/2+\varepsilon}} \|u\|_{X_{100\varepsilon, 1/2+\varepsilon}}^2 + \dots \\ &+ T^\varepsilon \|\nabla u_1\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|\nabla u_2\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|u_3\|_{X_{100\varepsilon, 1/2+\varepsilon}} + \dots \end{aligned}$$

We further have

$$\begin{aligned} \|\nabla u_1\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|\nabla u_2\|_{X_{100\varepsilon, 1/2+\varepsilon}} &\lesssim \sum_{M, N} \|S_M \nabla u_1\|_{X_{100\varepsilon, 1/2+\varepsilon}} \|S_N \nabla u_2\|_{X_{100\varepsilon, 1/2+\varepsilon}} \lesssim \\ &\lesssim \sum_{M \geq N} \frac{N}{M} \|S_M u\|_{X_{2+100\varepsilon, 1/2+\varepsilon}} \|S_N u\|_{X_{100\varepsilon, 1/2+\varepsilon}} \lesssim \|u\|_{X_{2+100\varepsilon, 1/2+\varepsilon}} \|u\|_{X_{100\varepsilon, 1/2+\varepsilon}}. \end{aligned}$$

Inserting the estimates in (60), we get

$$\|u\|_{X_{2+100\varepsilon, 1/2+\varepsilon}} \lesssim \|u_0\|_{H^{2+100\varepsilon}} + T^\varepsilon \|u\|_{X_{2+100\varepsilon, 1/2+\varepsilon}} \|u\|_{X_{100\varepsilon, 1/2+\varepsilon}}^2$$

Choosing T with $T^\varepsilon \|u\|_{X_{100\varepsilon, 1/2+\varepsilon}}^2 \lesssim T^\varepsilon \|u_0\|_{H^{100\varepsilon}}^2 \ll 1$ allows us to hide again. We obtain

$$\|u\|_{X_{2+100\varepsilon, 1/2+\varepsilon}} \lesssim \|u_0\|_{H^{2+100\varepsilon}}.$$

In a similar fashion, one obtains the estimates for every $k = 100\varepsilon + n$. We omit the details. \square

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